On inequalities of Hermite-Hadamard type for co-ordinated \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convex functions

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Abstract

In the paper, the authors establish some integral inequalities of Hermite-Hadamard type for co-ordinated \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convex functions on a rectangle of the first quadrant in a plane.

Keywords: \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convex function; co-ordinates; rectangle of the plane; Hermite-Hadamard type inequality

1. Introduction

The following definitions are known in the literature.

**Definition 1.1.** A function \(f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}\) is said to be convex if \(f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\) holds for all \(x, y \in I\) and \(\lambda \in [0, 1]\).

**Definition 1.2** ([7]). For \(f : [0, b] \rightarrow \mathbb{R}\) and \(m \in (0, 1]\), if \(f(\lambda x + m(1-\lambda)y) \leq \lambda f(x) + m(1-\lambda)f(y)\) is valid for all \(x, y \in [0, b]\) and \(\lambda \in [0, 1]\), then we say that \(f(x)\) is an \(m\)-convex function on \([0, b]\).

**Definition 1.3** ([6]). For \(f : [0, b] \rightarrow \mathbb{R}\) and \((\alpha, m) \in (0, 1] \times (0, 1]\), if \(f(\lambda x + m(1-\lambda)y) \leq \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y)\) is valid for all \(x, y \in [0, b]\) and \(\lambda \in [0, 1]\), then we say that \(f(x)\) is an \((\alpha, m)\)-convex function on \([0, b]\).

**Definition 1.4** ([3, 4]). A function \(f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}\) is said to be convex on the co-ordinates on \(\Delta\) if the partial mappings \(f_u : u \in [a, b] \rightarrow f(u, y) \in \mathbb{R}\) and \(f_v : v \in [c, d] \rightarrow f(x, v) \in \mathbb{R}\) are convex for all \(x \in (a, b)\) and \(y \in (c, d)\).

**Definition 1.5.** [3, 4] A function \(f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}\) is said to be convex on the co-ordinates on \(\Delta\) if

\[
f(tx + (1-t)z, \lambda y + (1-\lambda)w) \leq t\lambda f(x, y) + t(1-\lambda)f(x, w) + (1-t)(1-\lambda)f(z, y) + (1-t)\lambda f(z, w)
\]

holds for all \(t, \lambda \in [0, 1]\) and \((x, y), (z, w) \in \Delta\).

We now recall some inequalities of Hermite-Hadamard type.

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Theorem 1.1 ([5]). Let \( f : [0, \infty) \to \mathbb{R} \) be \( m \)-convex and \( m \in (0, 1] \). If \( f \in L_1([a, b]) \) for \( 0 \leq a < b < \infty \), then
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf(b/m)}{2}, \frac{m f(a/m) + f(b)}{2} \right\}.
\]

Theorem 1.2 ([3, Theorem 2.2]). Let \( f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \) be convex on the co-ordinates on \( \Delta \). Then we have
\[
f\left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b \frac{1}{d-c} \int_c^d \frac{f(x, y) \, dy}{d(x, y)} \, dx \right] \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b \int_c^d f(x, y) \, dy \, dx \right]
\]
\[
\leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)].
\]

In [1, 2], the authors introduced the following co-ordinated \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convex functions.

Definition 1.6 ([1, 2]). For \( m_1, m_2, \alpha_1, \alpha_2 \in (0, 1] \), a function \( f : [0, b] \times [0, d] \to \mathbb{R} \) is said to be co-ordinated \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convex if
\[
f(tx + m_1(1-t)z, \lambda y + m_2(1-\lambda)w)
\]
\[
\leq t^{\alpha_1} \lambda^{\alpha_2} f(x, y) + m_1(1-\lambda^{\alpha_2})f(x, z) + m_2(1-\lambda^{\alpha_1}) f(x, w) + m_1 m_2 (1-\lambda^{\alpha_2}) (1-\lambda^{\alpha_1}) f(z, w) \quad (1.1)
\]
holds for all \((t, \lambda) \in [0, 1] \times [0, 1) \) and \((x, y), (z, w) \in [0, b] \times [0, d]\).

For more information on Hermite-Hadamard type inequalities for various kinds of convex functions, please refer to the monograph [4], recently published papers [1, 2, 8, 9], and closely related references therein.

In this paper, we will establish some integral inequalities of Hermite-Hadamard type for co-ordinated \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convex functions.

2. Integral inequalities of Hermite-Hadamard type

Theorem 2.1. Let \( f : \left[ 0, \frac{b}{m_1^2} \right] \times \left[ 0, \frac{d}{m_2^2} \right] \to \mathbb{R} \) be an integrable function with \( 0 \leq a < b \) and \( 0 \leq c < d \) for some fixed \( m_1, m_2 \in (0, 1] \). If \( f \) is co-ordinated \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convex on \( \left[ 0, \frac{b}{m_1^2} \right] \times \left[ 0, \frac{d}{m_2^2} \right] \) for \( \alpha_1, \alpha_2 \in (0, 1] \), then
\[
f\left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2^{\alpha_1 + \alpha_2 + 1}} \left[ \frac{1}{b-a} \int_a^b G(x, \frac{c + d}{2}) \, dx \right] \leq \frac{1}{2^{\alpha_1 + \alpha_2}} \int_a^b \int_c^d \left[ G(x, \frac{y}{m_1}) + m_1(2^{\alpha_1} - 1)G\left( \frac{x}{m_1}, \frac{y}{m_2} \right) \right] \, dy \, dx
\]
where
\[
G(u, v) = f(u, v) + m_1(2^{\alpha_1} - 1)f\left( \frac{u}{m_1}, \frac{v}{m_2} \right) + m_2(2^{\alpha_2} - 1)f\left( u, \frac{v}{m_2} \right) + m_1 m_2 (2^{\alpha_1} - 1)(2^{\alpha_2} - 1)f\left( \frac{u}{m_1}, \frac{v}{m_2} \right)
\]
for \((u, v) \in [a, \frac{b}{m_1^2}] \times [c, \frac{d}{m_2^2}]\).

Proof. Using the \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convexity of \( f \) with \( t = \lambda = \frac{1}{2} \) in (1.1), we have
\[
f\left( \frac{a + b}{2}, \frac{c + d}{2} \right) = \int_0^1 f\left( \frac{ta + (1-t)b + (1-t)a + tb}{2}, \frac{c + d}{2} \right) \, dt
\]
\[
\leq \int_0^1 \left[ \frac{1}{2^{\alpha_1 + \alpha_2}} f\left( \frac{ta + (1-t)b}{2}, \frac{c + d}{2} \right) \right] \leq m_1 \left( 1 - \frac{1}{2^{\alpha_1}} \right) \frac{1}{2^{\alpha_2}} f\left( \frac{(1-t)a + tb}{m_1}, \frac{c + d}{2} \right) + m_2 \frac{1}{2^{\alpha_1}} \left( 1 - \frac{1}{2^{\alpha_2}} \right)
\]
\[
\times f\left( \frac{ta + (1-t)b}{2}, \frac{c + d}{2} \right) + m_1 m_2 \left( 1 - \frac{1}{2^{\alpha_2}} \right) f\left( \frac{(1-t)a + tb}{m_1}, \frac{c + d}{2} \right) \, dt
\]
\[
= \frac{1}{2^{\alpha_1 + \alpha_2}} \int_a^b \left[ f\left( x, \frac{c + d}{2} \right) \right] \, dx
\]
for \((x, y) \in [0, b] \times [c, d]\\)}.
Utilizing the last two inequalities leads to the inequality (2.1). The proof of Theorem 2.1 is complete.

Similarly, we have

\[ f(x, \frac{c + d}{2}) = \int_0^1 f \left( x, \frac{\lambda c + (1 - \lambda)d + (1 - \lambda)c + \lambda d}{2} \right) d\lambda \]

Taking \( y = \lambda c + (1 - \lambda)d \) for \( 0 \leq \lambda \leq 1 \) and using the \((\alpha_1, m_1)-(\alpha_2, m_2)\)-convexity of \( f \) with \( t = \lambda = \frac{1}{2} \) in (1.1) give

\[
f \left( \frac{x, c + d}{2} \right) = \int_0^1 f \left( x, \frac{\lambda c + (1 - \lambda)d + (1 - \lambda)c + \lambda d}{2} \right) \frac{d\lambda}{2}
\]

Combining the above inequalities arrives at

\[
\begin{align*}
f \left( \frac{x, m_1, m_2}{2} \right) & \leq \frac{1}{2^{(\alpha_1 + \alpha_2)(d - c)}} \int_a^b G \left( \frac{x, m_1, m_2}{2} \right) d\lambda, \\
f \left( \frac{x, m_1, m_2}{2} \right) & \leq \frac{1}{2^{(\alpha_1 + \alpha_2)(d - c)}} \int_a^b G \left( \frac{x, m_1, m_2}{2} \right) d\lambda,
\end{align*}
\]

and

\[
f \left( \frac{x, m_1, m_2}{2} \right) \leq \frac{1}{2^{(\alpha_1 + \alpha_2)(d - c)}} \int_a^b G \left( \frac{x, m_1, m_2}{2} \right) d\lambda.
\]

Combining the above inequalities arrives at

\[
f \left( \frac{a + b, c + d}{2} \right) \leq \frac{1}{2^{(\alpha_1 + \alpha_2)(d - c)}} \int_a^b G \left( \frac{a + b, c + d}{2} \right) dx
\]

\[
\leq \frac{1}{2^{(\alpha_1 + \alpha_2)(d - c)}} \int_a^b \int_c^d \left[ G(x, y) + m_1(2^{\alpha_1} - 1)G \left( \frac{x, m_1, y}{m_2} \right) + m_2(2^{\alpha_2} - 1)G \left( \frac{x, y, m_1}{m_2} \right) \right] dy dx.
\]

Similarly, we have

\[
f \left( \frac{a + b, c + d}{2} \right) \leq \frac{1}{2^{(\alpha_1 + \alpha_2)(d - c)}} \int_a^b G \left( \frac{a + b, c + d}{2} \right) dy
\]

\[
\leq \frac{1}{2^{(\alpha_1 + \alpha_2)(d - c)}} \int_a^b \int_c^d \left[ G(x, y) + m_1(2^{\alpha_1} - 1)G \left( \frac{x, m_1, y}{m_2} \right) + m_2(2^{\alpha_2} - 1)G \left( \frac{x, y, m_1}{m_2} \right) \right] dx dy.
\]

Utilizing the last two inequalities leads to the inequality (2.1). The proof of Theorem 2.1 is complete.

**Corollary 2.1.1.** Under the conditions of Theorem 2.1,

1. if \( \alpha_1 = \alpha_2 = \alpha \) and \( m_1 = m_2 = m \), then

\[
f \left( \frac{a + b, c + d}{2} \right) \leq \frac{1}{2^{(\alpha + 1)}} \int_a^b G \left( \frac{a + b, c + d}{2} \right) dx + \frac{1}{d - c} \int_c^d G \left( \frac{a + b, y}{2} \right) dy
\]

\[
\leq \frac{1}{16^{(d - c)}} \int_a^b \int_c^d \left[ G(x, y) + m(2^{\alpha} - 1)G \left( \frac{x, y}{m} \right) + m^2(2^{\alpha} - 1)^2G \left( \frac{x, y}{m} \right) \right] dx dy.
\]

2. if \( \alpha_1 = \alpha_2 = 1 \) and \( m_1 = m_2 = m \), then

\[
f \left( \frac{a + b, c + d}{2} \right) \leq \frac{1}{8} \int_a^b G \left( \frac{a + b, c + d}{2} \right) dx + \frac{1}{d - c} \int_c^d G \left( \frac{a + b, y}{2} \right) dy
\]

\[
\leq \frac{1}{16(d - c)} \int_a^b \int_c^d \left[ G(x, y) + mG \left( \frac{x, y}{m} \right) + m^2G \left( \frac{x, y}{m} \right) \right] dx dy.
\]
Theorem 2.2. Let \( f : [0, b/m_1] \times [0, d/m_2] \to \mathbb{R} \) be an integrable function with \( 0 \leq a < b \) and \( 0 \leq c < d \) for some fixed \( m_1, m_2 \in (0, 1] \). If \( f \) is co-ordinated \((\alpha_1, m_1)\)-\((\alpha_2, m_2)\)-convex on \( [0, b/m_1] \times [0, d/m_2] \) for some fixed \( \alpha_1, \alpha_2 \in (0, 1] \), then

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy \leq \frac{1}{2^{\alpha_1 + 1}(\alpha_2 + 1)(b-a)} \int_a^b M(x, c, d) \, dx + \frac{1}{2^{\alpha_2 + 1}(\alpha_1 + 1)(d-c)} \int_c^d N(a, b, y) \, dy
\]

and

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy \leq \frac{1}{2^{\alpha_1 + \alpha_2 + 1}(\alpha_1 + 1)(\alpha_2 + 1)} \left[ 2N(a, b, c) + m_1(2^{\alpha_1} + 1)N\left( \frac{a}{m_1}, b \right) + m_1(2^{\alpha_2} + 1)N\left( \frac{a}{m_1}, d \right) + m_2(2^{\alpha_1} + \alpha_1 - 1)N\left( \frac{a}{m_1}, \frac{b}{m_2} \right) \right]
\]

where

\( M(x, w, z) = f(x, w) + m_1(2^{\alpha_1} - 1)f\left( \frac{x}{m_1}, z \right) + m_2(2^{\alpha_2} - 1)f\left( \frac{x}{m_2}, \frac{z}{m_2} \right) \)

and

\( N(u, v, y) = f(u, y) + m_1(2^{\alpha_1} - 1)f\left( \frac{u}{m_1}, y \right) + m_1(2^{\alpha_2} - 1)f\left( \frac{u}{m_2}, \frac{y}{m_2} \right) \)

for \( u, v, y \in [a, \frac{b}{m_1}] \) and \( x, w, z \in [c, \frac{d}{m_2}] \).

Proof. Setting \( y = \lambda c + (1 - \lambda)d \) for \( 0 < \lambda < 1 \). From the \((\alpha_1, m_1)\)-\((\alpha_2, m_2)\)-convexity of \( f \) with \( t = \frac{1}{2} \) and \( 0 < \lambda < 1 \) in (1.1), we obtain

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy \leq \frac{1}{b-a} \int_0^1 \int_a^b f(x, \lambda c + (1 - \lambda)d) \, dx \, d\lambda
\]

and

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy \leq \frac{1}{b-a} \int_a^b M(x, c, d) \, dx.
\]

Obviously, if letting \( x = ta + (1-t)b \) for \( 0 \leq t \leq 1 \) and using the \((\alpha_1, m_1)\)-\((\alpha_2, m_2)\)-convexity of \( f \) with \( 0 \leq t \leq 1 \) and \( \lambda = \frac{1}{2} \) in (1.1), it follows that

\[
\frac{1}{b-a} \int_a^b f(x, c) \, dx \leq \frac{1}{2^{\alpha_1}} \int_0^1 \left\{ \lambda^{\alpha_1} f(a, c) + m_1(1 - t^{\alpha_1})f\left( \frac{b}{m_1}, c \right) + m_2(2^{\alpha_1} - 1)f\left( \frac{a}{m_1}, \frac{c}{m_2} \right) + m_1m_2(1 - t^{\alpha_1})(2^{\alpha_2} - 1)f\left( \frac{b}{m_1}, \frac{c}{m_2} \right) \right\} \, dt
\]

and

\[
\frac{1}{b-a} \int_a^b f\left( \frac{x}{m_1}, c \right) \, dx \leq \frac{1}{2^{\alpha_1} \alpha_2 + 1} N\left( \frac{a}{m_1}, \frac{b}{m_1}, c \right), \quad \frac{1}{b-a} \int_a^b f\left( x, \frac{d}{m_2} \right) \, dx \leq \frac{1}{2^{\alpha_2}(\alpha_1 + 1)} N\left( \frac{a}{m_1}, \frac{d}{m_2}, \right),
\]

\[
\frac{1}{b-a} \int_a^b f\left( \frac{x}{m_1}, \frac{d}{m_2} \right) \, dx \leq \frac{1}{2^{\alpha_2} \alpha_2 + 1} N\left( \frac{a}{m_1}, \frac{b}{m_1}, \right).
\]

When combining the above inequalities, we find

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy \leq \frac{1}{2^{\alpha_1}(\alpha_2 + 1)(b-a)} \int_a^b M(x, c, d) \, dx
\]

and

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy \leq \frac{1}{2^{\alpha_1 + \alpha_2}(\alpha_1 + 1)(\alpha_2 + 1)} \left[ N(a, b, c) + m_1(2^{\alpha_1} - 1)N\left( \frac{a}{m_1}, \frac{b}{m_1}, c \right) + m_2(2^{\alpha_1} + \alpha_1 - 1)N\left( \frac{a}{m_1}, \frac{d}{m_2} \right) + m_1m_2(2^{\alpha_2} - 1)N\left( \frac{a}{m_1}, \frac{b}{m_1}, \frac{d}{m_2} \right) \right]
\]

Similarly, we have

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy \leq \frac{1}{2^{\alpha_2}(\alpha_1 + 1)(d-c)} \int_c^d N(a, b, y) \, dy
\]
Combining the last two inequalities leads to the inequality in Theorem 2.2. The proof of Theorem 2.2 is thus complete.

Corollary 2.2.1. Under the conditions of Theorem 2.2, if \( \alpha_1 = \alpha_2 = \alpha \) and \( m_1 = m_2 = m \), then
\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dx \, dy \leq \frac{1}{2^{\alpha+1} + \alpha} \int_a^b \int_c^d [f(x,y) + (2^\alpha - 1)f(b,y)] \, dx \, dy
\]
\[
\leq \frac{2^{\alpha+1} - 1}{2^{2\alpha+1} + \alpha^2} \left[ 2N(a,b,c) + m(2^\alpha + \alpha - 1)N \left( \frac{a}{m}, \frac{b}{m}, \frac{d}{m} \right) \right]
\]
\[+ m(2^\alpha + \alpha - 1)N \left( \frac{a}{m}, \frac{b}{m}, \frac{d}{m} \right) \]

Corollary 2.2.2. Under the assumptions of Theorems 2.1 and 2.2, if \( \alpha_1 = \alpha_2 = \alpha \) and \( m_1 = m_2 = m \), then
\[
f \left( \frac{a+b+c+d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( \frac{x, c+d}{2} \right) \, dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy \right]
\]
\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dx \, dy
\]
\[
\leq \frac{1}{2^{\alpha+1}} \left[ \frac{1}{b-a} \int_a^b \left[ f(x,c) + (2^\alpha - 1)f(x,d) \right] \, dx + \frac{1}{d-c} \int_c^d \left[ f(a,y) + (2^\alpha - 1)f(b,y) \right] \, dy \right]
\]
\[
\leq \frac{2^{\alpha+1} - 1}{2^{2\alpha+1} + \alpha^2} \left[ 2N(a,b,c) + m(2^\alpha + \alpha - 1)N \left( \frac{a}{m}, \frac{b}{m}, \frac{d}{m} \right) \right]
\]
\[+ m(2^\alpha + \alpha - 1)N \left( \frac{a}{m}, \frac{b}{m}, \frac{d}{m} \right) \]

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