Global Journal of Mathematical Analysis, 3 (2) (2015) 89-96
www.sciencepubco.com/index.php/GJMA
©Science Publishing Corporation
doi: 10.14419/gjma.v3i2.4655
Research Paper

The greatest prime function of real variable

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Abstract

In this paper, we introduce a new function $f(x)=[x]_p, x \in R$, with $x \geq 2$, which we call the greatest prime function. In addition, we give an extension of the function to R, and then use this definition to prove some inequalities and properties of this function. Some illustrative examples are given.

Keywords: *Greatest Prime Function; Gap Function; Inequalities.*

1. Introduction

In this paper, we introduce a new function

$$f(x)=[x]_p, x \in R, \text{ with } x \geq 2,$$

Which we will call the greatest prime function (or the floor prime function), where $[x]_p \left(or \ \lfloor x \rfloor_p\right)$ represents the greatest prime number less than or equal to the real number example, $[2.99]_p = 2$, $[\sqrt{8}]_p = 2$, $[7]_p = 7, [10.87]_p = 7$, $[\sqrt[3]{9578643}]_p = 211$, etc.

The domain of the greatest prime function is $R \setminus (-\infty, 2)$ and its range is the set of all prime numbers. Note that the graph of the greatest prime function jumps one unit at the prime number 3, two units at 5, two units at 7, and four units at 11, and so on. This implies that the function is discontinuous at each prime number. The graph of $f(x)=[x]_p$, is shown in Fig.1.

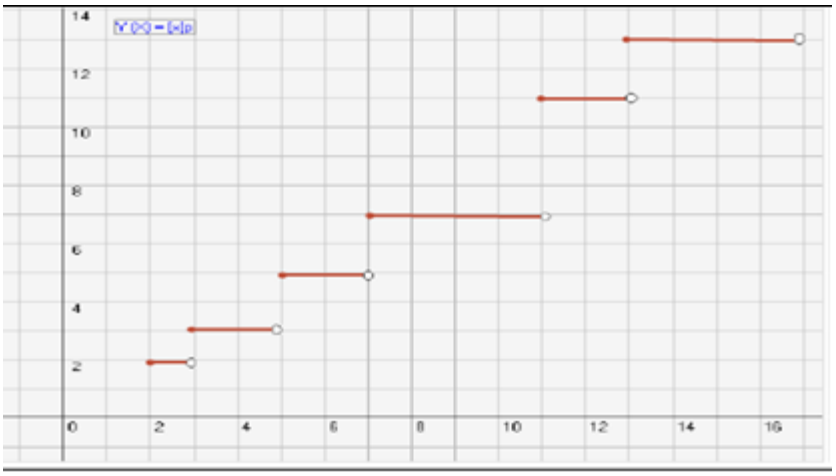


Fig. 1

Here, we also define a fractional gap function by $\{x\}_p = x - [x]_p, \forall x \geq 2$, which means the non-integer gap number. The graph of $f(x) = x - [x]_p$ is shown in Fig.2.

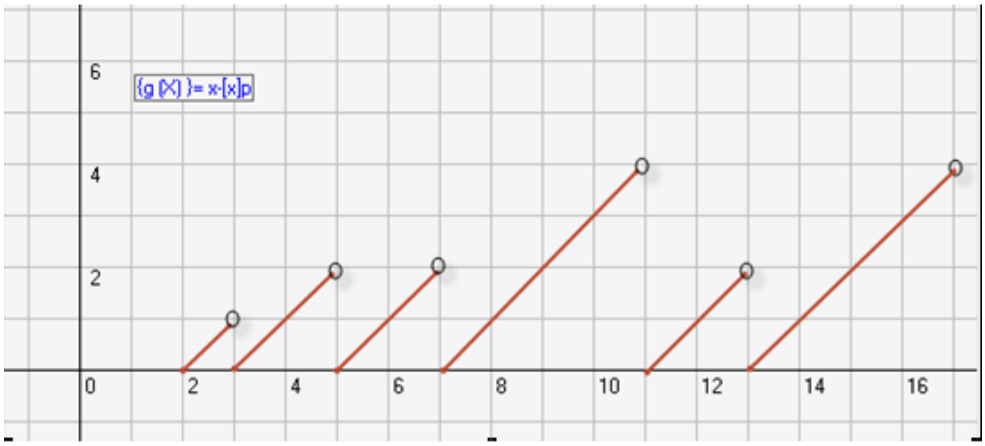


Fig. 2

For example, if $x = (7479089197619)^{1/2}$ then

$$\{x\}_p = \{(7479089197619)^{1/2}\} = (7479089197619)^{1/2} - [(7479089197619)^{1/2}]_p$$
$$= 2734792.35 - 2734763 = 29.35$$

It should be noted that the first prime number next to the prime number 2734763 is 2734793, hence the gap between them is 30, which is greater than the non-integer gap 29.35. In general, it follows from the definition of greatest prime function that $\{x\}_p < g_n$, where $g_n = p_{n+1} - p_n$, and $p_n \leq x$. Indeed, let $n \geq 1$ be an integer, and let $p_n \geq 2$ denote the nth prime number. Then, since $p_n \leq x < p_{n+1}$, we obtain $\{x\}_p = x - [x]_p = x - p_n < p_{n+1} - p_n$.

In this paper, we also extend the greatest prime function to R in an odd manner,

$$G_p(x) = \begin{cases} [x]_p, & x \geq 2 \\ 0, & |x| < 2 \\ -[-x]_p, & x \leq -2 \end{cases} \tag{1}$$

Indeed, the function $G_p(x)$ defined by (1) is an odd function:

$$G_p(-x) = \begin{cases} [-x]_p, & -x \geq 2 \\ 0, & |x| < 2 \\ -[x]_p, & -x \leq -2 \end{cases}$$
$$= \begin{cases} -[x]_p, & x \geq 2 \\ 0, & |x| < 2 \\ [-x]_p, & x \leq -2 \end{cases}$$
$$= -G_p(x), \quad \forall x \in R.$$

For instance,

$$G_p(\sqrt[5]{9842178956}) = [99.68]_p = 97, \quad G_p(-\sqrt[5]{9842178956})$$
$$= [-99.68]_p = -[99.68]_p = -97 = -G_p(\sqrt[5]{9842178956}).$$

Remark 1.1: Note that one can make an extension for the greatest prime function to R , in even manner, so that

$$G_p(x)=\begin{cases}[x]_p\ , & x\geq 2\\ 0\ , & |x|<2\\ [-x]_p, & x\leq -2\end{cases}$$

Remark 1.2: We can similarly define the least prime function (or the ceiling prime function), and denote it by $\lceil x\rceil_p$, where $\lceil x\rceil_p$ represents the smallest prime number greater than or equal to the real number x. For example,

$$\lceil 2\rceil_p=2,\ \lceil 2.01\rceil_p=3,\ \left\lceil\sqrt{10.65}\right\rceil_p=5,$$

and so on.

Properties of the Greatest Prime Function

1) The function $[x]_p=x$ if and only if x is a prime number.
2) The function $[x]_p=p_n, n\in \mathrm{Z}^+$, if and only if $p_n\leq x<p_{n+1}$, where p_n is the nth prime number.
3) The function $[x]_p\leq x$, for any real number x .
4) The function $f(x)=[x]_p$ is nondecreasing for all $x\in R\setminus(-\infty,\ 2)$, i.e., if $x_1\leq x_2$, then $[x_1]_p\leq[x_2]_p$.
5) The function $f(x)=[x]_p$ is discontinuous at all primes $p_{n+1}, n\in \mathrm{Z}^+$ and continuous at all x satisfying $p_n\leq x<p_{n+1}$, because $[x]_p=p_n$, for any $n\in \mathrm{Z}^+$, with $n\geq 1$.
6) The function $f(x)=[x]_p$ is right-continuous at all primes p_n , because $\lim\limits_{x\to p_n+}[x]_p=p_n=f(p_n)=[p_n]$, for any $n\in \mathrm{Z}^+, n\geq 2$.

Example 1.1 Find the discontinuity points of $f(x)=x[x]_p, x\in R.$

We find that $x[x]_p=p_nx$, where $p_n\leq x<p_{n+1}, n\in Z^+$. Since $f(p_n)=p_n^2=\lim\limits_{x\to p_n+}p_nx$, and $\lim\limits_{x\to p_n-}x[x]_p=p_{n-1}p_n$, then $f(x)=[x]_p$ is discontinuous at the points $x=p_n$, for all integers $n\geq 2$.

Example 1.2 Consider the function

$$f(x)=\begin{cases}x\left[\frac{1}{x}\right]_p\ , & 0<x\leq\frac{1}{2}\\ 1\ , & x=0\end{cases}.$$

The function $f(x)$ is continuous in all intervals $\frac{1}{p_{n+1}}<x\leq\frac{1}{p_n}$, for any $n\in \mathrm{Z}^+, n\geq 1$. Moreover,

$$f\left(\frac{1}{p_n}\right)=1,\ f\left(\frac{1}{p_n}+0\right)=\lim_{x\to\frac{1}{p_n}+0}x\left[\frac{1}{x}\right]_p=\frac{p_{n-1}}{p_n}.$$

Therefore, the function is discontinuous at the points $x=\frac{1}{p_n}, n\in \mathrm{Z}^+$, with $n\geq 2$. Now, we will show that $f(x)$ is continuous at $x=0$. Indeed, For $x\in\left(\frac{1}{p_{n+1}},\ \frac{1}{p_n}\right)$, we have

$$\frac{p_n}{p_{n+1}}\leq x\left[\frac{1}{x}\right]_p<\frac{p_{n+1}}{p_n},\ n\in \mathrm{Z}^+.$$

By the prime number theorem (see [4]), we have $\lim\limits_{n\to\infty}\frac{p_{n+1}}{p_n}=1$. Therefore, since as $n\to\infty, x\to 0+$, we obtain

$$\lim_{x\to 0+}f(x)=\lim_{x\to 0+}x\left[\frac{1}{x}\right]_p=1=f(0).$$

Properties of the Fractional Semi-gap Function

1) The fractional semi-gap function $\{x\}_p=0$ if and only if x is a prime number ,i.e. $x=[x]_p$.

2) The function $0 \le \{x\}_p < g_n$, for all $x \in R, n \in Z^+$, where $g_n = p_{n+1} - p_n, n \ge 1$.

3) The fractional semi-gap function $\{x\}_p$ is discontinuous at all prime $p_{n+1}, n \in Z^+$ since

$$\lim_{x \to p_{n+1}^-} \{x\}_p = \lim_{x \to p_{n+1}^-} (x - [x]_p) = p_{n+1} - p_n,$$

and

$$\lim_{x \to p_{n+1}^+} \{x\}_p = \lim_{x \to p_{n+1}^+} (x - [x]_p) = \{p_{n+1}\}_p = 0.$$

4) The function $\{x\}_p$ is right-continuous at all x satisfying $p_n \le x < p_{n+1}$, for any $n \in Z^+$, with $n \ge 1$.

5) The function $\{x\}_p$ is continuous at all primes p_n, because $\lim_{x \to p_n^+} \{x\}_p = 0 = \{p_n\}_p$, for any $n \in Z^+, n \ge 2$. Now, we will give the statement of an interesting conjecture that was first stated in 1982 by the Iranian mathematician Farideh Firoozbakht [3], which we will use to establish some inequalities.

Firoozbakht's Conjecture: If p_k is the kth prime, then the sequence $(p_k)^{1/k}$ is strictly decreasing. Equivalently, for all $k \ge 1$ we have $p_{k+1}^k < p_k^{k+1}$.
The following result follows from the Firoozbakht's Conjecture.

Corollary1.1 If p_k is the kth prime, then the for all $k \ge 1$ we have $p_{k+1} < p_k^2$.

Proof. It is easily seen that if Firoozbakht's Conjecture is true, then for all $k \ge 1$, we have $p_{k+1} < p_k^{\frac{k+1}{k}} \le p_k^2$.

Remark 1.3: It should be noted here that the Corollary1.1 can be obtained from Bertrand-Chebyshev theorem, see [2]. Indeed, since $p_k \ge 2$, we have $p_{k+1} < 2p_k \le p_k^2$, for all $k \ge 1$.

Lemma 1.1: ***The General Classical Arithmetic-Geometric Mean Inequality***

If $x_1, x_2, ..., x_n$ are positive real numbers, then

$$(x_1 x_2 ... x_n)^{1/n} \le \frac{x_1 + x_2 + ... + x_n}{n},$$

with equality occurring if and only if $x_1 = x_2 = ... = x_n$.

2. Inequalities for the greatest prime function

In this section we prove several inequalities concerning the greatest prime function $[x]_p$.

Theorem 2.1: *For any integer $k \ge 1$, and for any real number $x \ge 2$, the greatest prime function satisfies the inequality*

$x < [x]_p^{\frac{k+1}{k}}$, where $[x]_p = p_k$.

Proof. Let p_k be the kth prime number, and let $x \in R$, $x \ge 2$ such that $x = p_k + r$, where $r < p_{k+1} - p_k$ and $k \ge 1$ is an integer. Then

$[x]_p = [p_k + r]_p > p_{k+1}^{\frac{k}{k+1}} > x^{\frac{k}{k+1}}$, which implies that $x < [x]_p^{\frac{k+1}{k}}$.
This proves the Theorem.
Now, we also establish some new interesting inequalities related to well-known inequalities in mathematics. First, we prove the following inequality for the primes $[x_1]_p, [x_2]_p, ..., [x_n]_p$, which is similar to a well-known general arithmetic-geometric mean inequality. So, we call it the general arithmetic-semigeometric mean inequality.

Theorem 2.2: ***The General Arithmetic-Semi geometric Mean Inequality***

If $x_1, x_2, ..., x_n$ are real numbers with $x_k \geq 2,$ for $1 \leq k \leq n,$ then

$$\left(\left[x_1 x_2 ... x_n\right]_p\right)^{\frac{1}{2n}} < \frac{\left[x_1\right]_p + \left[x_2\right]_p + ... + \left[x_n\right]_p}{n}.$$

Proof. Applying Corollary1.1, let us estimate

$$\left(\left[x_1 x_2 ... x_n\right]_p\right)^{\frac{1}{2n}} \leq \left(x_1 x_2 ... x_n\right)^{\frac{1}{2n}} < \left(\left[x_1\right]_p^2 \cdot \left[x_2\right]_p^2 ... \left[x_n\right]_p^2\right)^{\frac{1}{2n}}$$
$$= \left(\left[x_1\right]_p \cdot \left[x_2\right]_p ... \left[x_n\right]_p\right)^{\frac{1}{n}},$$

from which it follows that

$$\left(\left[x_1 . x_2 ... x_n\right]_p\right)^{\frac{1}{2n}} < \left(\left[x_1\right]_p \cdot \left[x_2\right]_p ... \left[x_n\right]_p\right)^{\frac{1}{n}}.$$

By using the classical arithmetic-geometric mean inequality for the prime numbers $\left[x_1\right]_p$, $\left[x_2\right]_p$,..., $\left[x_n\right]_p$, we get

$$\left(\left[x_1 x_2 ... x_n\right]_p\right)^{\frac{1}{2n}} < \frac{\left[x_1\right]_p + \left[x_2\right]_p + ... + \left[x_n\right]_p}{n},$$

which completes the proof.

Corollary 2.1 If $x_1, x_2, ..., x_n$ are real numbers with $x_k \geq 2,$ for $1 \leq k \leq n,$ then

$$\left(\left[x_1 x_2 ... x_n\right]_p\right)^{\frac{1}{n}} < \frac{\left[x_1\right]_p^2 + \left[x_2\right]_p^2 + ... + \left[x_n\right]_p^2}{n}.$$

Proof. This follows from the General Arithmetic-Semigeometric Mean Inequality, and from the inequality [see 1] :

$$\frac{\left[x_1\right]_p + \left[x_2\right]_p + ... + \left[x_n\right]_p}{n} \leq \sqrt{\frac{\left[x_1\right]_p^2 + \left[x_2\right]_p^2 + ... + \left[x_n\right]_p^2}{n}}.$$

To improve the previous estimation obtained in the Theorem 2.2, we establish the following inequality.

Theorem 2.3: ***The General Semi-arithmetic-Geometric Mean Inequality***

If $x_1, x_2, ..., x_n$ are real numbers with $x_k \geq 2,$ for $1 \leq k \leq n,$ then

$$\left(\left[x_1 x_2 ... x_n\right]_p\right)^{\frac{1}{n}} < \frac{\left[x_1\right]_p + \left[x_2\right]_p + ... + \left[x_n\right]_p}{n/2}$$

Proof. Using Bertrand-Chebyshev theorem, we obtain

$x < p_{k+1} < 2p_k = 2\left[x\right]_p$ for any real x.
Therefore, we get the estimate

$$\left(\left[x_1.x_2...x_n\right]_p\right)^{\frac{1}{n}} \le \left(x_1.x_2...x_n\right)^{\frac{1}{n}} < \left(2^n\left(\left[x_1\right]_p.\left[x_2\right]_p...\left[x_n\right]_p\right)\right)^{\frac{1}{n}}$$
$$= 2\left(\left[x_1\right]_p.\left[x_2\right]_p...\left[x_n\right]_p\right)^{\frac{1}{n}},$$

from which it follows that

$$\left(\left[x_1.x_2...x_n\right]_p\right)^{\frac{1}{n}} < 2\left(\left[x_1\right]_p.\left[x_2\right]_p...\left[x_n\right]_p\right)^{\frac{1}{n}}.$$

Now, using the classical arithmetic-geometric mean inequality for the prime numbers $\left[x_1\right]_p$, $\left[x_2\right]_p$,..., $\left[x_n\right]_p$, we get

$$\left(\left[x_1.x_2...x_n\right]_p\right)^{\frac{1}{n}} < \frac{\left[x_1\right]_p+\left[x_2\right]_p+...+\left[x_n\right]_p}{n/2},$$

which completes the proof.

Example 2.1 Consider the numbers $x_1=489,\ x_2=\sqrt{675},\ x_3=\sqrt[3]{8745},\ x_4=17\pi.$
We find that

$$\left[x_1.x_2.x_3.x_4\right]_p^{1/8} = [489.\sqrt{675}.\sqrt[3]{8745}.17\pi]_p^{1/8} = [822300.503]_p^{1/8} = \left(822299\right)^{1/8} = 5.487$$

And

$$\frac{\left[489\right]_p+\left[\sqrt{675}\right]_p+\left[\sqrt[3]{8745}\right]_p+\left[17\pi\right]_p}{4} = \frac{487+23+19+53}{4} = \frac{582}{4} = 145.5$$

Example 2.2 Let $x_1=3.9,\ x_2=3.99,\ x_3=3.999$.
For the prime number $\left[x_1.x_2.x_3\right]_p$, we have

$$\left[x_1.x_2.x_3\right]_p = [3.9\cdot 3.99\cdot 3.999]_p = [62.228]_p = 61,$$

and

$$\left(\frac{\left[3.9\right]_p+\left[3.99\right]_p+\left[3.999\right]_p}{3}\right)^6 = \left(\frac{3+3+3}{3}\right)^6 = 729.$$

Now, we use the semi-arithmetic-geometric mean inequality of type to estimate the number $\left[x_1.x_2.x_3\right]_p$.
We have

$$61 = \left[x_1.x_2.x_3\right]_p < \left(\frac{\left[3.9\right]_p+\left[3.99\right]_p+\left[3.999\right]_p}{3/2}\right)^3 = \left(\frac{18}{3}\right)^3 = 216.$$

Theorem 2.4: Triangle Inequality of Type I If $x_1,\ x_2,...,x_n$ are real numbers with $x_k \ge 2$, for $1 \le k \le n$, then

$$\left(\left[x_1+x_2+...+x_n\right]_p\right)^{1/2} < \left[x_1\right]_p+\left[x_2\right]_p+...+\left[x_n\right]_p. \quad (2)$$

Proof. Using Corollary1.1, we get the estimation

$$\left[x_1+x_2+...+x_n\right]_p \le x_1+x_2+...+x_n < \left[x_1\right]_p^2+\left[x_2\right]_p^2+...+\left[x_n\right]_p^2 \le \left(\left[x_1\right]_p+\left[x_2\right]_p+...+\left[x_n\right]_p\right)^2.$$

Taking the second root in the previous inequality, we get (2).

Example 2.3 Consider the real numbers $x_1=\sqrt{189},\ x_2=\sqrt{275},\ x_3=\sqrt[3]{9347}$.
We find that

$$\left(\left[x_1+x_2+x_3\right]_p\right)^{1/2}=\left(\left[\sqrt{189}+\sqrt{275}+\sqrt[3]{9347}\right]_p\right)^{1/2}=\left([51.395]_p\right)^{1/2}=47^{1/2}=6.855,$$

And

$$\left[x_1\right]_p+\left[x_2\right]_p+\left[x_3\right]_p=\left[\sqrt{189}\right]_p+\left[\sqrt{275}\right]_p+\left[\sqrt[3]{9347}\right]_p=13+13+19=45.$$

Now, we will show that using Bertrand-Chebyshev theorem we can improve the previous estimation in the following inequality.

Theorem 2.5: Triangle Inequality of Type II If $x_1,x_2,...,x_n$ are real numbers with $x_k \ge 2$, for $1\le k \le n$, then

$$\frac{1}{2}\left[x_1+x_2+...+x_n\right]_p < \left(\left[x_1\right]_p+\left[x_2\right]_p+...+\left[x_n\right]_p\right).$$

Proof. For any real number x , there is $k \in Z^+$ such that $x=p_k+r_k$, where $p_k=\left[x\right]_p$, $r_k < p_{k+1}-p_k$.
Using Bertrand-Chebyshev theorem we obtain $x < p_{k+1} < 2p_k = 2\left[x\right]_p$, for any real x.
Then, we get the estimation

$$\left[x_1+x_2+...+x_n\right]_p \le x_1+x_2+...+x_n < 2\left(\left[x_1\right]_p+\left[x_2\right]_p+...+\left[x_n\right]_p\right).$$

Example 2.4 Let $x_1=\sqrt{189}, x_2=\sqrt{275},\ x_3=\sqrt[3]{9347}$.
First, Let us estimate the prime number $\left[x_1+x_2+x_3\right]_p$ using the triangle Inequality of Theorem 2.4

$$\left[x_1+x_2+x_3\right]_p=\left[\sqrt{189}+\sqrt{275}+\sqrt[3]{9347}\right]_p=\left([51.395]_p\right)=47,$$

and

$$\left(\left[x_1\right]_p+\left[x_2\right]_p+\left[x_3\right]_p\right)^2=\left(\left[\sqrt{189}\right]_p+\left[\sqrt{275}\right]_p+\left[\sqrt[3]{9347}\right]_p\right)^2=(13+13+19)^2=2025.$$

Now, we use the triangle inequality Theorem 2.5 to estimate the prime number $\left[x_1+x_2+x_3\right]_p$, and obtain

$$\left[x_1+x_2+x_3\right]_p=[51.395]_p=47.$$

On the other hand, we have

$$2\left(\left[x_1\right]_p+\left[x_2\right]_p+\left[x_3\right]_p\right)=2\left(\left[\sqrt{189}\right]_p+\left[\sqrt{275}\right]_p+\left[\sqrt[3]{9347}\right]_p\right)=2(13+13+19)=90.$$

Theorem 2.6: ***Inequality of Bernoulli Type***. If x is a real number with $x \geq 2$, then

$$\left(1+[x]_p\right)^n \geq 1+nx, \text{ for any integer } n \geq 2, \tag{3}$$

where $[x]_p$ is the greatest prime number of real x, and $[x]_p = p_s$ is sth prime number.
Proof. Let $[x]_p$ be any prime number, and let $x = p_s + r_s$, where $r_s < p_{s+1} - p_s$.
The proof utilizes mathematical induction. The case $n = 2$ yields inequality

$$\left(1+[x]_p\right)^2 = \left(1+[p_s + r]_p\right)^2 = \left(1+p_s\right)^2 = 1+2p_s + p_s^2.$$

Using the Corollary 1.1, and Bertrand-Chebyshev theorem we get

$$\left(1+[x]_p\right)^2 > 1+2p_s + p_s^2 > 1+2p_{s+1} > 1+2x,$$

So that the assertion is valid in this case. Thus, we assume the validity of the inequality (3) for a positive n, and shall deduce it for $n+1$. The assumption of $\left(1+[x]_p\right)^n > 1+nx$, and the fact that $1+[x]_p > 0$ implies that

$$\begin{aligned}\left(1+[x]_p\right)^{n+1} &> \left(1+nx\right)\left(1+[x]_p\right) = 1+nx+[x]_p\left(1+nx\right)\\ &> 1+nx+nx[x]_p\\ &\geq 1+nx+2nx, \text{ since } [x]_p \geq 2.\\ &> 1+nx+x = 1+(n+1)x.\end{aligned}$$

Thus, the inequality (3) for $n+1$ follows. Hence the inequality is true for all integers $n \geq 2$.

Example 2.5 Let $x_1 = \sqrt{5}$. Then $\left(1+[\sqrt{5}]_p\right)^2 = 3^2 = 9 \geq 1+2*\sqrt{5} = 5.472$.

3. Conclusion

Here, we present a new function $f(x) = [x]_p, x \in R$, with $x \geq 2$, which we call the greatest prime function. Because our results could be applied in the theory of prime numbers and other fields of science, they are likely to be of great interest to the researchers.

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