Logarithmically complete monotonicity of a power-exponential function involving the logarithmic and psi functions

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Abstract

Let $\Gamma$ and $\psi = \frac{\Gamma'}{\Gamma}$ be respectively the classical Euler gamma function and the psi function and let $\gamma = -\psi(1) = 0.57721566\ldots$ stand for the Euler-Mascheroni constant. In the paper, the authors simply confirm the logarithmically complete monotonicity of the power-exponential function $q(t) = t^{\psi(t) - \ln t - \gamma}$ on the unit interval $(0, 1)$, concisely deny that $q(t)$ is a Stieltjes function, surely point out fatal errors appeared in the paper [V. Krasniqi and A. Sh. Shabani, On a conjecture of a logarithmically completely monotonic function, Aust. J. Math. Anal. Appl. 11 (2014), no. 1, Art. 5, 5 pages; Available online at http://ajmaa.org/cgi-bin/paper.pl?string=v11n1/V1111PS.tex], and partially solve a conjecture posed in the article [B.-N. Guo, Y.-J. Zhang, and F. Qi, Refinements and sharpenings of some double inequalities for bounding the gamma function, J. Inequal. Pure Appl. Math. 9 (2008), no. 1, Art. 17; Available online at http://www.emis.de/journals/JIPAM/article953.html].

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1. Introduction

For bounding the classical Euler gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0$$

and for bounding the ratio $\frac{\Gamma(x)}{\Gamma(y)}$, Guo, Zhang, and Qi established some inequalities in [10]. When comparing two inequalities, they created the function

$$q(t) = t^{\psi(t) - \ln t - \gamma}, \quad t > 0$$

(1.1)
in [10, Remark 8], where \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \) is called the psi function and \( \gamma = 0.577 \ldots \) stands for the Euler-Mascheroni constant. As said in [10, Remark 8], Guo, Zhang, and Qi demonstrated, by using the software Mathematica to plot, the decreasing monotonicity of \( q(t) \) on \((0, \infty)\). Then they conjectured that the function \( q(t) \) is logarithmically completely monotonic on \((0, \infty)\). A function \( f \) is said to be logarithmically completely monotonic on an interval \( I \) if it is infinitely differentiable and satisfies
\[
(-1)^k \ln f(t))^{(k)} \geq 0
\]
on \( I \) for \( k \in \mathbb{N} \), where \( \mathbb{N} \) denotes the set of positive integers. See \[2, 3, 6, 13, 14, 16, 17, 19, 20\] and plenty of closely related references in \[18\]. For our own convenience, we denote the set of logarithmically completely monotonic functions on \( I \) by \( \mathcal{L}[I] \).

Recall from \[12, Chapter XIII\], \[20, Chapter 1\], and \[21, Chapter IV\] that a function \( f(x) \) is said to be completely monotonic on an interval \( I \) if and only if \( f \) has derivatives of all orders and satisfies
\[
(-1)^{k-1} f^{(k-1)}(x) \geq 0
\]
on \( I \) for \( k \in \mathbb{N} \). The class of completely monotonic functions on \((0, \infty)\) may be characterized by \[21, p. 161, Theorem 12b\] which reads that a necessary and sufficient condition that \( f(x) \) should be completely monotonic for \( 0 < x < \infty \) is that
\[
f(x) = \int_0^\infty e^{-xs} \, d\mu(s),
\]
where \( \mu(s) \) is non-decreasing and the integral converges for \( 0 < x < \infty \). We will use \( \mathcal{C}[I] \) to denote the set of completely monotonic functions on \( I \).

Also recall from \[3, 20, 21\] that if \( f(x) \) can be represented as
\[
f(x) = a + \int_0^\infty \frac{1}{x + s} \, d\mu(s)
\]
on \((0, \infty)\), where \( a \geq 0 \), the measure \( \mu \) is nonnegative on \([0, \infty)\), and
\[
\int_0^\infty \frac{1}{1 + s} \, d\mu(s) < \infty,
\]
then \( f(x) \) is said to be a Stieltjes function. The set of Stieltjes functions will be denoted by \( \mathcal{S} \).

In \[11\], Krasniqi and Shabani claimed that they confirmed the above conjecture. However, because \(-\ln t \in \mathcal{C}([0, 1])\), but \(-\ln t \notin \mathcal{C}([0, \infty])\), all the proofs of \[11, Conjecture 2.1 and Theorem 2.2\] are wrong.

Our main results in this paper may be stated as the following theorem.

**Theorem 1.1.** The function \( q(t) \) defined by (1.1) satisfies
\[
q(t) \in \mathcal{L}([0, 1]), \quad q(t) \notin \mathcal{S}, \quad \lim_{t \to 0^+} q(t) = \infty, \quad \lim_{t \to \infty} q(t) = 0.
\]

It is clear that Theorem 1.1 partially confirms the logarithmically complete monotonicity of the function \( q(t) \) on \((0, \infty)\) and partially solves the above conjecture posed in \[10, Remark 8\].

### 2. Proof of theorem 1.1

In \[3, 5, 6, 13, 14, 19\], the inclusions
\[
\mathcal{S} \setminus \{0\} \subset \mathcal{L}([0, \infty]) \quad \text{and} \quad \mathcal{L}[I] \subset \mathcal{C}[I]
\]
were discovered. For more information, see \[18\] and plenty of closely related references therein. In order to prove the first property in (1.2), from the first inclusion in (2.1), it suffices to show that the function \( q(t) \) is a Stieltjes function. In \[3\], Berg remarked that \( f \in \mathcal{S} \) and \( f \neq 0 \) if and only if \( \frac{1}{xf(x)} \in \mathcal{S} \). Therefore, if \( q(t) \in \mathcal{S} \), then \( \frac{1}{tq(t)} \notin \mathcal{S} \), and so \( \frac{1}{tq(t)} \) should be decreasing on \((0, \infty)\). However, an easy calculation leads to
\[
1 = \left. \frac{1}{tq(t)} \right|_{t=1} < \left. \frac{1}{tq(t)} \right|_{t=2} = 2^{3(\gamma - 1) + 2 \ln 2} = 1.08518525 \ldots
\]
As a result, the function \( q(t) \) is not a Stieltjes function, that is, the second property in (1.2) is true.

In \[1, pp. 374–375, Theorem 1\] and \[9, p. 105, Theorem 1\], Alzer, Guo, and Qi proved that
\[
\theta_\alpha(x) = x^\alpha [\ln x - \psi(x)] \in \mathcal{C}([0, \infty])
\]
if and only if \( \alpha \leq 1 \) and that

\[
\lim_{x \to 0^+} \theta_1(x) = 1, \quad \lim_{x \to \infty} \theta_1(x) = \frac{1}{2}, \quad \lim_{x \to 0^+} \theta_\alpha(x) = \infty, \quad \lim_{x \to \infty} \theta_\alpha(x) = 0, \quad \alpha < 1.
\] (2.2)

It is clear that

\[
\ln q(t) = \{t[\psi(t) - \ln t] - \gamma\} \ln t = (\ln t)[\theta_1(t) + \gamma].
\] (2.3)

Since \(-\ln t \in C([0,1])\) and the product or sum of finitely many completely monotonic functions is still a completely monotonic function on their common interval, it is easy to see that \(\ln q(t) \in C([0,1])\) and \(q(t) \in L([0,1])\).

The limits in (1.2) follows immediately from taking \(t \to 0^+\) and \(t \to \infty\) on both sides of the equation (2.3) and making use of the first two limits in (2.2).

The first limit in (1.2) may also be deduced directly from the argument

\[
\lim_{t \to 0^+} \ln q(t) = \left\{ \lim_{t \to 0^+} [t[\psi(t)] - \ln t - \gamma] \right\} \lim_{t \to 0^+} \ln t
= \left\{ \lim_{t \to 0^+} [t[\psi(t)] + 1 - 1 - \gamma] \right\} \lim_{t \to 0^+} \ln t
= \left\{ \lim_{t \to 0^+} \left[ \psi(t) + \frac{1}{t} \right] - 1 - \gamma \right\} \lim_{t \to 0^+} \ln t
= \left\{ \lim_{t \to 0^+} [t[\psi(t + 1)] - 1 - \gamma] \right\} \lim_{t \to 0^+} \ln t
= -(1 + \gamma) \lim_{t \to 0^+} \ln t
= \infty.
\]

The proof of Theorem 1.1 is complete.

3. Remarks

Remark 3.1. On the interval \((1, \infty)\), it is obvious that the function \(\ln t\) is positive and increasing and \(\theta_1(t) + \gamma \in C([1, \infty])\).

Therefore, we can not make clear the monotonicity of the functions \((\ln t)[\theta_1(t) + \gamma] = -\ln q(t)\) and \((-\ln t)[\theta_1(t) + \gamma] = \ln q(t)\), say nothing of the logarithmically complete monotonicity of \(q(t)\), on \((1, \infty)\). This is the key difficulty to completely solve the above conjecture.

Remark 3.2. By a similar argument to the second property in (1.2), we can show that \(1 - G(x) \not\in S\), where

\[
G(x) = \left[ 1 - \frac{\ln x}{\ln(x + 1)} \right] x \ln x, \quad x > 0.
\]

However, Guo and Qi conjectured that \(1 - G(x) \in C([0, \infty])\) in [8] and its preprint [15]. Later, Berg and Pedersen verified this conjecture in [4].

4. An open problem

For \(\beta \in \mathbb{R}\), let

\[
h^\beta(t) = t^{|\psi(t)-\ln t|-\beta}, \quad t > 0.
\]

If \(h^\beta(t) \in L([0, \infty])\), then the first derivative of its logarithm

\[
\left[\ln h^\beta(t)\right]' = \frac{t^2 \psi'(t) \ln t - t(\ln t)^2 - 2t \ln t + t \psi(t) + t \psi(t) \ln t - b}{t}
\]

should be non-positive on \((0, \infty)\), that is,

\[
b \geq t^2 (\ln t) \psi'(t) + t(1 + \ln t) \psi(t) - (\ln t + 2)t \ln t \to -\frac{1}{2}, \quad t \to \infty.
\]

This means that \(\beta \geq -\frac{1}{2}\) is a necessary condition such that \(h^\beta(t) \in L([0, \infty])\).

Is the condition \(\beta \geq -\frac{1}{2}\) sufficient for \(h^\beta(t) \in L([0, \infty])\)?

Open Problem 4.1. Determine the best constant \(-\frac{1}{2} \leq \beta \leq \gamma\) such that \(h^\beta(t) \in L([0, \infty])\).

Remark 4.1. This paper is a slightly revised version of the preprint [7].
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References


