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Research paper

Descending Periods in Compositions of Entire Functions

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Abstract

We consider a question of A. and C. Rényi concerning entire functions f and g for which $f \circ g$ is periodic. We isolate the case in which a period translation of $f \circ g$ descends through g to a conformal automorphism of $g(\mathbb{C})$. In this descending regime we obtain a complete classification: either g is periodic, or

$g(z+T) = g(z) + K$

for some nonzero constant K, and f is K-periodic. Consequently, any example outside the periodic and translation cases must come from a non-descending period. The quadratic pullback examples lie in this remaining case.

Keywords: *Composition of entire functions; Deck transformations; Entire functions; Periodic functions; Rényi problem*

1. Introduction

Let f and g be entire functions. A question attributed to A. and C. Rényi asks what can be said about g if $f \circ g$ is periodic. The problem goes back to Rényi and Rényi [25] and has recently been revisited by Eremenko [1]. The classical literature already contains several results on periodic entire functions and periodic compositions; see, for example, Baker [26], Gross [27], and Gross and Yang [28]. Rényi and Rényi proved their assertion under the additional assumption that either f or g is a polynomial [25], while Eremenko's note also records an unpublished proof of the real entire case by Gleizer and an announcement by Gaida [1, 29].
The standard examples may be summarized as follows:

1. g is periodic;
2. $g(z+T) = g(z) + K$, and f is K-periodic;
3. g is a quadratic polynomial, for instance $g(z) = z^2$ and $f(z) = \cos\sqrt{z}$;
4. more generally, $g = P \circ h$, where P is quadratic and $h(z+T) = h(z) + K$, while $f \circ P$ is K-periodic.

These examples are conjectured to describe the general situation, up to minor modifications.
There has been substantial recent activity on periodicity questions for entire and meromorphic functions. In particular, extensions of Rényi-type periodicity results and of Yang-type periodicity problems have been developed by Latreuch and Zemirni [2], Liu, Korhonen and Liu [3], Liu and Wei [4], and Zemirni, Laine and Latreuch [7]. These works study conditions under which periodicity of polynomial, differential, or differential–difference expressions forces periodicity of an underlying entire function. They provide a modern context for the Rényi problem, although the composition problem considered here has a different geometric feature: the interaction between a period of $f \circ g$ and the fibers of the inner function g.
A second closely related direction concerns shifts and difference operators. The identity $f(g(z+T)) = f(g(z))$ naturally compares the two translated values $g(z+T)$ and $g(z)$, so recent uniqueness and value-sharing results for functions and their shifts are relevant background. Developments in this direction include work of He, Xiao and Fang [5], Barki, Dyavanal and Bhoosnurmath [6], Ahamed [8], Qiu and Qi [9], and Wang and Chen [10]. Further results on higher differences, differential–difference operators and small-function sharing were obtained by Mallick and Basak [11], Wang and Chen [12], Miloudi [13], Qin, Long and Wang [14], He, Fang and Xiao [15], Wang, He and Fang [16], and Chaithra, Jayarama and Naveenkumar [17]. Although these papers do not all address compositions directly, they describe the current analytic framework for translation identities, difference operators, and periodicity phenomena.
The subject has continued to develop in 2025–2026. Recent contributions include uniqueness results for shifts and differential polynomials by Wang and Chen [18], for derivatives and difference operators by Saha and Mallick [19], and for entire functions sharing values with

their shifts by Miloudi [20] and Xu, Lin and Chen [21]. Related extensions to difference or shift operators were obtained by Maity and Biswas [22], Pramanik and Dey [23], and Zhang and Chen [24]. Taken together, this recent literature shows that translations of entire or meromorphic functions carry strong rigidity when they are compatible with additional analytic structure. The present paper isolates a different, geometric rigidity mechanism: a period translation of the composite is assumed to descend through the inner function to an automorphism of its image.

We therefore study the case in which the period translation of $f\circ g$ is compatible with the inner function g. Suppose that $h=f\circ g$ has period $T\neq 0$. Then

$$f(g(z+T))=f(g(z)).$$

Thus the two functions $g(z+T)$ and $g(z)$ have the same image under f. In general this does not imply that $g(z+T)$ is a single-valued function of $g(z)$; the quadratic example $g(z)=z^2$ is of this type. We consider precisely the case where the period translation does descend through g.

Definition 1.1. *Let g be a nonconstant entire function and let $T\neq 0$. We say that the translation $z\mapsto z+T$ descends through g if there is a conformal automorphism Φ of the Riemann surface $g(\mathbb{C})$ such that*

$$g(z+T)=\Phi(g(z))\qquad (z\in\mathbb{C}).$$

By Picard's theorem, $g(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C}\setminus\{a\}$. The automorphism groups of these two surfaces are explicit, which yields the following classification.

Theorem 1.2 (Classification in the descending case)**.** *Let f and g be nonconstant entire functions, and suppose that*

$$h=f\circ g$$

has period $T\neq 0$. Assume that the translation $z\mapsto z+T$ descends through g to a conformal automorphism of $g(\mathbb{C})$. Then one of the following holds:

1. *g is periodic;*
2. *there exists $K\neq 0$ such that*

   $$g(z+T)=g(z)+K$$

   for all z, and f is K-periodic.

Thus a nonperiodic example which is not of translation type must come from a period which does not descend to an automorphism of $g(\mathbb{C})$. The quadratic examples have this feature.

In this sense the descending regime gives no further examples: the automorphism induced on $g(\mathbb{C})$ is either a translation, giving the second alternative, or has finite order, giving periodicity of g.

2. Automorphisms preserving an entire function

We first record two elementary facts about entire functions invariant under automorphisms of $\mathbb{C}$ or $\mathbb{C}^*$.

Lemma 2.1. *Let f be a nonconstant entire function, and let*

$$\Phi(w)=aw+b,\qquad a\neq 0,$$

be an affine automorphism of $\mathbb{C}$. Suppose

$$f(\Phi(w))=f(w)\qquad (w\in\mathbb{C}).$$

Then either

1. *$a=1$ and b is a period of f, or*
2. *$a\neq 1$, a is a root of unity, and Φ has finite order.*

Proof. If $a=1$, then $f(w+b)=f(w)$. Thus b is a period of f, allowing the case $b=0$.

Suppose $a\neq 1$. Let $c=b/(1-a)$, so that c is the fixed point of Φ. Put $F(u)=f(u+c)$. Then

$$F(au)=F(u).$$

Writing $F(u)=\sum_{n=0}^{\infty}c_nu^n$, we obtain

$$c_n(a^n-1)=0\qquad (n\geq 0).$$

Since F is nonconstant, some $c_n\neq 0$ with $n\geq 1$, and therefore $a^n=1$. Hence a is a root of unity. It follows that Φ has finite order. □

Lemma 2.2. *Let f be a nonconstant entire function. Let Φ be an automorphism of $\mathbb{C}^*$, and suppose*

$$f(\Phi(w))=f(w)\qquad (w\in\mathbb{C}^*).$$

Then $\Phi(w)=aw$, where a is a root of unity. In particular, Φ has finite order.

Proof. Every automorphism of $\mathbb{C}^*$ has the form $\Phi(w)=aw$ or $\Phi(w)=a/w$, with $a\neq 0$.

The second possibility cannot occur. If $f(a/w)=f(w)$ on $\mathbb{C}^*$, then $f(a/w)$ has a removable singularity at $w=0$. Equivalently, $f(\zeta)$ has a finite limit as $\zeta\to\infty$. Thus f is bounded outside a disk, and hence f is constant, a contradiction.

Therefore $\Phi(w)=aw$. The identity $f(aw)=f(w)$, initially on $\mathbb{C}^*$, extends to $w=0$. Writing

$$f(w)=\sum_{n=0}^{\infty}c_nw^n,$$

we get $c_n(a^n-1)=0$ for every n. Since f is nonconstant, $a^n=1$ for some $n\geq 1$. Hence a is a root of unity. □

3. Proof of the descending-period theorem

Proof of Theorem 1.2. Let

$\Omega = g(\mathbb{C})$.

By Picard's theorem, $\Omega = \mathbb{C}$ or $\Omega = \mathbb{C} \setminus \{a\}$ for some $a \in \mathbb{C}$. By assumption, there is a conformal automorphism Φ of Ω such that

$g(z+T) = \Phi(g(z))$.

Since $f(g(z+T)) = f(g(z))$, we have

$f(\Phi(w)) = f(w) \qquad (w \in \Omega)$.

First suppose $\Omega = \mathbb{C}$. Then $\Phi(w) = aw+b$. By Lemma 2.1, either Φ is a translation or Φ has finite order. If Φ has finite order m, then

$g(z+mT) = \Phi^m(g(z)) = g(z)$,

so g is periodic. If $\Phi(w) = w+K$, then

$g(z+T) = g(z)+K$.

If $K=0$, then g is T-periodic. If $K \neq 0$, the identity

$f(w+K) = f(w)$

shows that f is K-periodic.
Now suppose $\Omega = \mathbb{C} \setminus \{a\}$. Put

$F(u) = f(u+a), \qquad \Psi(u) = \Phi(u+a)-a$.

Then Ψ is an automorphism of $\mathbb{C}^*$, and

$F(\Psi(u)) = F(u) \qquad (u \in \mathbb{C}^*)$.

By Lemma 2.2, Ψ, and hence Φ, has finite order. Therefore, for some $m \geq 1$,

$g(z+mT) = \Phi^m(g(z)) = g(z)$.

Thus g is periodic. □

4. The non-descending case

Theorem 1.2 leaves the case in which $z \mapsto z+T$ does not descend through g. The non-descending regime encompasses the quadratic examples. Indeed, if $g(z) = z^2$, then $g(z+T)$ is not a single-valued function of $g(z)$. Nevertheless, for

$f(w) = \cos\sqrt{w}$

one has

$f(g(z)) = \cos z$,

which is periodic.
We emphasize that Theorem 1.2 does not classify all non-descending periods. The general non-descending case remains outside the scope of the present paper; the quadratic construction is an explicit family of such examples.
More generally, if P is a quadratic polynomial, h is entire and

$h(z+T) = h(z)+K$,

and if $f \circ P$ is K-periodic, then

$f(P(h(z+T))) = f(P(h(z)))$.

Thus $g = P \circ h$ gives a periodic composition.

Remark 4.1 (Polynomial pullbacks and the descending criterion)**.** *Suppose more generally that P is a polynomial, h is a nonconstant entire function satisfying*

$h(z+T) = h(z)+K, \qquad K \neq 0$,

and $g = P \circ h$. If the translation $z \mapsto z+T$ descended through g, then there would exist a conformal automorphism Φ of $g(\mathbb{C})$ such that

$P(h(z)+K) = \Phi(P(h(z)))$.

The relation $h(\mathbb{C})+K = h(\mathbb{C})$, together with Picard's theorem, forces $h(\mathbb{C}) = \mathbb{C}$: indeed, an omitted value would be translated to a different omitted value. Hence $g(\mathbb{C}) = P(\mathbb{C}) = \mathbb{C}$, so $\Phi(w) = aw+b$. The identity theorem gives

$P(w+K) = aP(w)+b \qquad (w \in \mathbb{C})$.

Comparison of leading coefficients yields $a = 1$. If $d = \deg P \geq 2$, however, the polynomial $P(w+K) - P(w)$ has degree $d-1$, with leading coefficient $da_dK \neq 0$, and therefore cannot be the constant b. Thus a polynomial pullback with $\deg P \geq 2$ and $K \neq 0$ cannot produce an additional descending structure. In degree one, one recovers exactly the translation alternative of Theorem 1.2.

The remaining part of the Rényi problem may be viewed as the problem of showing that non-descending periods arise only in this quadratic manner.

5. Conclusion

We proved that if the period translation of $f \circ g$ descends through g to an automorphism of $g(\mathbb{C})$, then the Rényi problem has only the periodic and translation-type outcomes. Thus the polynomial-quadratic examples occur outside the descending case. This separates the problem into two parts: the descending case, handled above, and the non-descending case, where the expected obstruction is quadratic. No classification of the full non-descending regime is claimed here.

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