

Existence and Uniqueness Results for ψ - α - θ Fuzzy Quasi-Controlled Hybrid Contractions with Applications to Integral, Fractional, and Directed Systems

John Pamba^{1*} and Isaac Daniel Tembo²

¹Department of ICT, Zambia University College of Technology, Ndola, Zambia.

²Department of Mathematics, Statistics and Actuarial Sciences, University of Zambia, Lusaka, Zambia.

*Corresponding author E-mail: jpamba@zut.edu.zm

Abstract

In this paper, we introduce a new class of ψ - α - θ fuzzy quasi-controlled hybrid contractions in the framework of controlled fuzzy quasi-metric spaces. The proposed model integrates fuzziness, asymmetry, and control mechanisms into a unified structure, thereby extending classical Banach and Kannan contraction principles as well as several recent developments in fuzzy fixed point theory. A fixed point theorem guaranteeing existence, uniqueness, and convergence of Picard iterations is established under admissibility and continuity assumptions. Unlike classical contraction frameworks, the proposed hybrid condition does not require a global Lipschitz constant less than one, but instead employs a combination of simulation, control, and hybrid aggregation functions to regulate nonlinear behavior. The effectiveness of the framework is illustrated through nontrivial examples and counterexamples, including cases where the mappings are neither Banach nor Kannan contractions and where classical fixed point theorems fail. These results demonstrate that the proposed approach constitutes a strict and nontrivial generalization of existing contraction principles. Applications to nonlinear integral equations, fractional differential equations, and directed systems are presented. In particular, we establish existence and uniqueness of solutions in settings where classical contraction conditions are violated, including non-expansive operators and cases with effective Lipschitz constants exceeding unity. The results provide a flexible and robust analytical framework for studying nonlinear problems in fuzzy and asymmetric environments, with potential applications in applied mathematics, engineering, and computational sciences.

Keywords: Fixed point theorem; Fractional Differential Equations; Fuzzy quasi-controlled metric space; Generalized contraction; Nonlinear integral equation; Picard iteration.

1. Introduction

The Banach contraction principle [1] remains one of the most fundamental results in nonlinear analysis, providing a powerful tool for establishing the existence and uniqueness of fixed points in complete metric spaces. Owing to its wide applicability in differential equations, integral equations, optimization, and dynamical systems, numerous generalizations have been developed to accommodate increasingly sophisticated mathematical structures. Among the most influential are b -metric spaces [2, 3], controlled metric spaces [4], and several generalized metric-type spaces in which the classical triangle inequality is replaced by more flexible control conditions.

The introduction of fuzzy set theory by Zadeh [5] provided an effective mathematical framework for modelling uncertainty and vagueness. Building on this foundation, Kramosil and Michálek [6] introduced fuzzy metric spaces, which were subsequently refined by George and Veeramani [7]. Since then, fuzzy fixed point theory has evolved into an active research area, producing numerous contraction principles and existence results for nonlinear operators under uncertainty [8, 9]. Recent developments have further incorporated admissibility conditions, simulation functions, and hybrid contraction mechanisms, considerably extending the applicability of fuzzy fixed point methods [10, 11, 12, 13].

Many mathematical models arising in optimization, decision sciences, information theory, network analysis, and directed dynamical systems possess an intrinsic directional structure that cannot be adequately represented within symmetric metric spaces. This limitation motivated the development of quasi-metric spaces by Kelly [14], where the symmetry axiom is removed. Their fuzzy counterparts, introduced by Gregori and Romaguera [15], provide an appropriate framework for modelling asymmetric similarity and directional uncertainty. More recently, generalized contraction principles have also been investigated in fuzzy b -metric spaces and related asymmetric settings [10, 16, 17].

Independently, controlled metric spaces [4] have proved highly successful in extending classical fixed point theory by incorporating control functions into the triangle inequality. This additional degree of flexibility has generated numerous extensions involving graph structures, generalized metric spaces, partial metric-type spaces, and nonlinear operators. Representative contributions include fixed point results in controlled partial metric-type spaces endowed with graphs [18], graph-based controlled metric spaces [19, 20], and Suzuki-type contraction principles in controlled settings [21]. Parallel developments in fuzzy analysis include controlled fuzzy metric spaces, fuzzy strong controlled metric spaces, fuzzy double-controlled metric spaces, and other generalized fuzzy metric structures [22, 23, 24, 25, 26, 27, 28]. These developments demonstrate the growing importance of control functions in extending fuzzy fixed point theory to increasingly general nonlinear settings. Furthermore, recent work has established structural and topological properties of fuzzy quasi-metric spaces, including continuity, completeness, and entropy-based invariants [29].

Beyond fixed point theory, recent investigations have demonstrated that simulation-function techniques are equally effective in best proximity point theory, thereby extending the applicability of fuzzy metric methods to problems where fixed points need not exist [30]. This development further illustrates the flexibility of simulation-function approaches in nonlinear analysis.

Despite these advances, the existing literature generally treats fuzziness, directional asymmetry, admissibility, simulation functions, and control mechanisms separately. While important contributions have combined some of these ingredients, a unified framework simultaneously incorporating controlled fuzzy quasi-metric structures, admissibility conditions, simulation functions, and hybrid aggregation operators appears to be largely absent. Consequently, several nonlinear mappings that naturally involve directional behaviour and multiple proximity measures remain outside the scope of existing contraction principles.

Motivated by this observation, we introduce the notion of a (ψ, α, θ) -fuzzy quasi-controlled hybrid contraction on controlled fuzzy quasi-metric spaces. The proposed framework combines four complementary components: an admissibility function governing the pairs on which the contraction is imposed, a simulation function controlling the strength of the contraction, an aggregation operator integrating several fuzzy proximity measures, and the underlying controlled fuzzy quasi-metric structure accommodating directional asymmetry. This formulation unifies several existing contraction models while permitting substantially greater flexibility in the analysis of nonlinear mappings.

The main contributions of this paper are summarized as follows.

1. We introduce the concept of controlled fuzzy quasi-metric spaces together with their fundamental structural properties and establish their relationship with controlled fuzzy metric spaces.
2. We define the class of (ψ, α, θ) -fuzzy quasi-controlled hybrid contractions and demonstrate that several well-known contraction principles arise as particular cases through appropriate choices of the defining functions.
3. We establish existence and uniqueness theorems for fixed points under suitable admissibility and continuity assumptions and prove the convergence of the associated Picard iteration.
4. We provide illustrative examples showing that the proposed framework applies to mappings that fail to satisfy classical Banach-type and Kannan-type contractive conditions.
5. We apply the theoretical results to nonlinear integral equations, fractional differential equations, and directed nonlinear systems by verifying the hypotheses of the main theorem within appropriate complete function spaces.

The remainder of the paper is organized as follows. Section 2 introduces the necessary preliminaries and basic properties of controlled fuzzy quasi-metric spaces. Section 3 develops the new (ψ, α, θ) -hybrid contraction framework and establishes the principal fixed point results. Section 4 presents applications to nonlinear integral equations, fractional differential equations, and directed nonlinear systems. Finally, Section 5 concludes the paper with remarks and directions for future research.

2. Preliminaries

In this section, we recall basic definitions and results that will be used throughout the paper. We begin with the notion of fuzzy sets and fuzzy metric spaces.

2.1. Fuzzy Metric Spaces

We define a fuzzy quasi-metric by generalising the definition of a fuzzy metric given by George and Veeramani in [7] as follows:

Definition 2.1.1. A GV-fuzzy quasi-metric on a non-empty set X is a pair $(M, \star)$ where $\star$ is a continuous t -norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

1. $M(x, y, t) > 0$;
2. $M(x, x, t) = 1$ for all $t > 0$, and if $x \neq y$ then $M(x, y, t) < 1$;
3. $M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s)$;
4. $M(x, y, -) : (0, \infty) \rightarrow [0, 1]$ is continuous.

The triple $(X, M, \star)$ such that X is a non-empty set and $(M, \star)$ is a fuzzy quasi-metric on X is called a GV-fuzzy quasi-metric space.

In this paper, we shall refer to a GV-fuzzy quasi-metric space as just a fuzzy quasi-metric space.

Note that if $(M, \star)$ is a fuzzy quasi-metric on X , then $(M^{-1}, \star)$ is also a fuzzy quasi-metric on X , where M^{-1} is the fuzzy set in $X^2 \times (0, \infty)$ defined by

$$M^{-1}(x, y, t) = M(y, x, t).$$

Define M^i by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$. Then $(M^i, \star)$ is a fuzzy metric on X .

We now present the notion of a controlled metric space.

2.2. Controlled Metric Spaces

The idea of introducing control functions to generalize fixed point conditions in metric-type spaces was proposed by Mlaiki et al. [4].

Definition 2.2.1 (Controlled metric space [4]). Let X be a non-empty set and let $\alpha : X \times X \rightarrow [1, \infty)$ be a control function. A function $d_\alpha : X \times X \rightarrow [0, \infty)$ is called a controlled metric on X if for all $x, y, z \in X$, the following conditions hold:

1. $d_\alpha(x, y) = 0$ if and only if $x = y$;
2. $d_\alpha(x, y) = d_\alpha(y, x)$;
3. $d_\alpha(x, z) \leq \alpha(x, y)d_\alpha(x, y) + \alpha(y, z)d_\alpha(y, z)$.

The pair (X, d_α) is called a controlled metric space.

We define convergence, Cauchy and completeness in controlled metric spaces according to [18] as follows:

Definition 2.2.2. Let (X, d_α) be a controlled metric space and $\{x_n\}_{n \leq 0}$ be a sequence in X . Then

1. the sequence $\{x_n\}$ converges to some x in X if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\alpha(x_n, x) < \varepsilon$ for all $n \geq N$. In this case we write $\lim_{n \rightarrow \infty} x_n = x$,
2. the sequence x_n is Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\alpha(x_m, x_n) < \varepsilon$ for all $m, n \geq N$,
3. the controlled metric space (X, d_α) is called complete if every Cauchy sequence is convergent.

2.3. Fuzzy Controlled Metric Space

Definition 2.3.1 ([27]). Let X be a non-empty set, $\lambda : X \times X \rightarrow [1, \infty)$, $*$ be a continuous t -norm, and M_λ be a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $a, c, d \in X$ and $s, t > 0$:

1. $M_\lambda(a, c, 0) = 0$,
2. $M_\lambda(a, c, t) = 1$ if and only if $a = c$,
3. $M_\lambda(a, c, t) = M_\lambda(c, a, t)$,
4. $M_\lambda(a, d, t+s) \geq M_\lambda\left(a, c, \frac{t}{\lambda(a, c)}\right) * M_\lambda\left(c, d, \frac{s}{\lambda(c, d)}\right)$,
5. $M_\lambda(a, c, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous.

Then the triple $(X, M_\lambda, *)$ is called a controlled fuzzy metric space, and M_λ is called a controlled fuzzy metric on X .

Remark 2.3.2 ([27]). If for all $a, c \in X$, $\lambda(a, c) = s \geq 1$, then $(X, M_\lambda, *)$ reduces to a fuzzy b -metric space. Hence, every fuzzy b -metric space is a controlled fuzzy metric space. However, in general, a controlled fuzzy metric space is not necessarily an extended fuzzy b -metric space, even when $\alpha = \lambda$.

Example 2.3.3 ([27]). Let $X = A \cup C$, where $A = [0, 1]$ and $C = \mathbb{N} \setminus \{0, 1\}$. Define $M_\lambda : X \times X \times [0, \infty) \rightarrow [0, 1]$ by

$$M_\lambda(a, c, t) = \begin{cases} 1, & \text{if } a = c, \\ \frac{t}{t+1} c, & \text{if } a \in A, c \in C, \\ \frac{t}{t+1} a, & \text{if } a \in C, c \in A, \\ \frac{t}{t+1}, & \text{otherwise.} \end{cases}$$

Let the continuous t -norm $*$ be defined by $t_1 * t_2 = t_1 \cdot t_2$. Define $\lambda : X \times X \rightarrow [1, \infty)$ by

$$\lambda(a, c) = \begin{cases} 1, & \text{if } a, c \in A, \\ \max\{a, c\}, & \text{otherwise.} \end{cases}$$

Then $(X, M_\lambda, *)$ is a controlled fuzzy metric space. See [27] for details.

2.4. Fuzzy Quasi Controlled Metric Space

We now present the concept of fuzzy quasi controlled metric spaces as our preliminary new result. This notion extends fuzzy quasi-metric spaces and fuzzy controlled metric spaces by incorporating a direction-dependent control mechanism, while allowing intrinsic non-symmetry.

Definition 2.4.1. Let X be a non-empty set and $*$ be a continuous t -norm. A controlled fuzzy quasi-metric on X is a triple $(M, \star, \phi)$, where $M : X^2 \times (0, \infty) \rightarrow [0, 1]$ is a fuzzy set and $\phi : X \times X \rightarrow [1, \infty)$ is a control function such that, for all $x, y, z \in X$ and $t, s > 0$, the following conditions hold:

1. $M(x, y, t) > 0$;
2. $M(x, x, t) = 1$ for all $t > 0$, and if $x \neq y$ then $M(x, y, t) < 1$;
3. $M(x, z, t+s) \geq M(x, y, t/\phi(x, y)) * M(y, z, s/\phi(y, z))$;
4. for each $x, y \in X$, the mapping $t \mapsto M(x, y, t)$ is continuous on $(0, \infty)$.

The quadruple $(X, M, \star, \phi)$ is called a controlled fuzzy quasi-metric space.

Remark 2.4.2. If $\phi(x, y) \equiv 1$ for all $x, y \in X$, then the above definition reduces to the classical fuzzy quasi-metric space in the sense of George and Veeramani.

Example 2.4.3. Let $X = \mathbb{R}$ and let $\star = \min$ be the Gödel continuous t -norm. Define the control function

$$\phi(x, y) = \begin{cases} 2, & x > y, \\ 1, & x \leq y, \end{cases}$$

and define $M : X \times X \times (0, \infty) \longrightarrow [0, 1]$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ e^{-(x-y)/t}, & x > y, \\ \frac{t}{t+y-x}, & x < y, \end{cases} \quad t > 0.$$

Then $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space.

Indeed, we verify each axiom of Definition 2.4.1.

- (i) Since $e^{-(x-y)/t} > 0$, $\frac{t}{t+y-x} > 0$, for every $t > 0$, it follows that $M(x, y, t) > 0$ for all $x, y \in X$.
(ii) For every $x \in X$, $M(x, x, t) = 1$. If $x \neq y$, then either

$$M(x, y, t) = e^{-(x-y)/t} < 1,$$

or

$$M(x, y, t) = \frac{t}{t+y-x} < 1.$$

Hence the second axiom is satisfied.

- (iii) Let $x, y, z \in X$ and $s, t > 0$.

Since $\star = \min$, it is enough to prove

$$M(x, z, t+s) \geq \min \left\{ M\left(x, y, \frac{t}{\phi(x, y)}\right), M\left(y, z, \frac{s}{\phi(y, z)}\right) \right\}.$$

The verification follows by considering the relative order of x, y, z .

If $x \leq y \leq z$, then

$$M(x, z, t+s) = \frac{t+s}{t+s+z-x},$$

while

$$M\left(x, y, \frac{t}{\phi(x, y)}\right) = \frac{t}{t+y-x},$$

because $\phi(x, y) = 1$, and similarly

$$M\left(y, z, \frac{s}{\phi(y, z)}\right) = \frac{s}{s+z-y}.$$

Since $z-x = (z-y) + (y-x)$, one has

$$\frac{t+s}{t+s+z-x} \geq \min \left\{ \frac{t}{t+y-x}, \frac{s}{s+z-y} \right\},$$

which proves the required inequality.

The remaining orderings $x \leq z < y$, $y < x \leq z$, $z < y < x$, etc., are verified similarly by substituting the corresponding expressions of M and the appropriate values of the control function ϕ . Hence the controlled triangle inequality holds in every case.

- (iv) For fixed x, y , both functions $t \mapsto e^{-(x-y)/t}$ and $t \mapsto \frac{t}{t+y-x}$ are continuous on $(0, \infty)$. Therefore $t \mapsto M(x, y, t)$ is continuous.

Consequently, $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space.

Finally, the space is not a controlled fuzzy metric since symmetry fails. Indeed, if $x > y$, then

$$M(x, y, t) = e^{-(x-y)/t},$$

whereas

$$M(y, x, t) = \frac{t}{t+x-y}.$$

For example, taking $x = 1$, $y = 0$ and $t = 1$,

$$M(1, 0, 1) = e^{-1}, \quad M(0, 1, 1) = \frac{1}{2},$$

and

$$e^{-1} \neq \frac{1}{2}.$$

Hence

$$M(x, y, t) \neq M(y, x, t),$$

so M is non-symmetric.

Therefore, $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space which is not a controlled fuzzy metric space.

Proposition 2.4.4. Let $(X, M, \star, \phi)$ be a controlled fuzzy quasi-metric space with $\phi(x, y) = \phi(y, x)$ for all $x, y \in X$. Define

$$M^{-1}(x, y, t) = M(y, x, t), \quad M^i(x, y, t) = \min\{M(x, y, t), M(y, x, t)\}.$$

Then $(X, M^{-1}, \star, \phi)$ is a controlled fuzzy quasi-metric space and $(X, M^i, \star, \phi)$ is a controlled fuzzy metric space.

Proof. We first show that $(X, M^{-1}, \star, \phi)$ is a controlled fuzzy quasi-metric space.

Since $M^{-1}(x, y, t) = M(y, x, t)$, all assertions follow directly from the corresponding properties of M .

(i) **Positivity.** Since $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space, $M(y, x, t) > 0$ for every $x, y \in X$ and $t > 0$. Hence

$$M^{-1}(x, y, t) > 0.$$

(ii) **Identity property.** For every $x \in X$,

$$M^{-1}(x, x, t) = M(x, x, t) = 1.$$

If $x \neq y$, then

$$M^{-1}(x, y, t) = M(y, x, t) < 1,$$

again by the defining properties of M .

(iii) **Controlled triangle inequality.** For arbitrary $x, y, z \in X$ and $t, s > 0$,

$$M^{-1}(x, z, t + s) = M(z, x, t + s).$$

Applying the controlled triangle inequality for M to the triple (z, y, x) yields

$$M(z, x, t + s) \geq M\left(z, y, \frac{t}{\phi(z, y)}\right) \star M\left(y, x, \frac{s}{\phi(y, x)}\right).$$

Since ϕ is symmetric,

$$\phi(z, y) = \phi(y, z), \quad \phi(y, x) = \phi(x, y),$$

we obtain

$$M^{-1}(x, z, t + s) \geq M^{-1}\left(x, y, \frac{s}{\phi(x, y)}\right) \star M^{-1}\left(y, z, \frac{t}{\phi(y, z)}\right).$$

Since the variables t and s are arbitrary positive numbers, interchanging their names gives

$$M^{-1}(x, z, t + s) \geq M^{-1}\left(x, y, \frac{t}{\phi(x, y)}\right) \star M^{-1}\left(y, z, \frac{s}{\phi(y, z)}\right),$$

which is precisely the controlled triangle inequality for M^{-1} .

(iv) **Continuity.** For fixed $x, y \in X$,

$$t \mapsto M^{-1}(x, y, t) = M(y, x, t)$$

is continuous because M is continuous in its third variable.

Hence $(X, M^{-1}, \star, \phi)$ is a controlled fuzzy quasi-metric space.

We now prove that $(X, M^i, \star, \phi)$ is a controlled fuzzy metric space.

Since

$$M^i(x, y, t) = \min\{M(x, y, t), M(y, x, t)\},$$

the symmetry property follows immediately:

$$\begin{aligned} M^i(y, x, t) &= \min\{M(y, x, t), M(x, y, t)\} \\ &= M^i(x, y, t). \end{aligned}$$

Thus M^i is symmetric.

The positivity and identity axioms are inherited from M because the minimum of two positive numbers is positive, and

$$M^i(x, x, t) = \min\{1, 1\} = 1.$$

If $x \neq y$, then both $M(x, y, t)$ and $M(y, x, t)$ are strictly less than one, so

$$M^i(x, y, t) < 1.$$

To verify the controlled triangle inequality, observe that

$$M^i(x, z, t + s) = \min\{M(x, z, t + s), M(z, x, t + s)\}.$$

Using the controlled triangle inequality for M ,

$$\begin{aligned} M(x, z, t + s) &\geq M\left(x, y, \frac{t}{\phi(x, y)}\right) \star M\left(y, z, \frac{s}{\phi(y, z)}\right), \\ M(z, x, t + s) &\geq M\left(z, y, \frac{t}{\phi(z, y)}\right) \star M\left(y, x, \frac{s}{\phi(y, x)}\right). \end{aligned}$$

Since ϕ is symmetric,

$$\phi(z, y) = \phi(y, z), \quad \phi(y, x) = \phi(x, y).$$

Therefore,

$$\begin{aligned} M^i(x, z, t + s) &\geq \min\left\{M\left(x, y, \frac{t}{\phi(x, y)}\right) \star M\left(y, z, \frac{s}{\phi(y, z)}\right), \right. \\ &\quad \left. M\left(y, x, \frac{s}{\phi(x, y)}\right) \star M\left(z, y, \frac{t}{\phi(y, z)}\right)\right\}. \end{aligned}$$

Since every continuous t -norm is monotone in each variable,

$$\min\{a \star b, c \star d\} \geq \min\{a, c\} \star \min\{b, d\},$$

for all $a, b, c, d \in [0, 1]$. Hence,

$$M^i(x, z, t + s) \geq M^i\left(x, y, \frac{t}{\phi(x, y)}\right) \star M^i\left(y, z, \frac{s}{\phi(y, z)}\right).$$

Thus the controlled triangle inequality holds.

Finally, since the minimum of two continuous functions is continuous,

$$t \mapsto M^i(x, y, t)$$

is continuous for every $x, y \in X$.

Consequently, $(X, M^i, \star, \phi)$ satisfies all the axioms of a controlled fuzzy metric space.

This completes the proof. □

Definition 2.4.5. [27] Let $(X, M_\lambda, \star)$ be a controlled fuzzy metric space. Then:

1. A sequence $\{a_n\}$ in X is said to be G -convergent to $a \in X$ if and only if

$$\lim_{n \rightarrow \infty} M_\lambda(a_n, a, t) = 1, \quad \text{for all } t > 0.$$

2. A sequence $\{a_n\}$ is said to be a G -Cauchy sequence if and only if

$$\lim_{n \rightarrow \infty} M_\lambda(a_n, a_{n+m}, t) = 1, \quad \text{for all } t > 0 \text{ and for each fixed } m \in \mathbb{N}.$$

3. The controlled fuzzy metric space $(X, M_\lambda, \star)$ is said to be G -complete if every G -Cauchy sequence in X is G -convergent in X .

Proposition 2.4.6. Let (x_n) be a sequence in X and $x \in X$. The following assertions are equivalent:

1. (x_n) left converges to x ;
2. $\lim_{n \rightarrow \infty} M(x, x_n, t) = 1$ for all $t > 0$;
3. For every $t > 0$, the sequence $M(x, x_n, t)$ converges to 1 uniformly on compact subsets of $(0, \infty)$.

Proof.

(1) $\Rightarrow$ (2) follows directly from the definition of left convergence.

(2) $\Rightarrow$ (3) follows from the continuity and monotonicity of the function $t \mapsto M(x, y, t)$.

(3) $\Rightarrow$ (1) is immediate. □

Remark 2.4.7. A sequence is said to be bi-convergent if it is both left and right convergent to the same limit. In general, bi-convergence is strictly stronger than either left or right convergence alone.

Definition 2.4.8 (Admissible Mapping). Let $(X, M, *, \beta)$ be a fuzzy quasi-controlled metric space. A mapping $T : X \rightarrow X$ is said to be α -admissible if there exists a function $\alpha : X \times X \rightarrow [0, \infty)$ such that for all $x, y \in X$, $\alpha(x, y) \geq 1$ and

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.$$

Definition 2.4.9 (Control Function Pair and Simulation Function, Azmi [23]). A triple (ψ, α, θ) is called a control-function system if

1. $\psi : [0, 1] \rightarrow [0, 1]$ is continuous, non-decreasing, and $\psi(t) = 1$ if and only if $t = 1$,
2. $\alpha : X \times X \rightarrow [0, \infty)$ satisfies the admissibility condition,
3. $\theta : [0, 1]^3 \rightarrow [0, 1]$ is continuous, non-decreasing in each variable, and $\theta(t, t, t) \leq t \quad \forall t \in [0, 1]$.

Such a system is used to define generalized fuzzy quasi-controlled hybrid contractions.

These preliminaries provide the foundation for defining our new ψ - α - θ fuzzy quasi-controlled hybrid contractions and proving the main fixed point results in the subsequent section.

3. Main Results

3.1. (ψ, α, θ) -Fuzzy Quasi-Controlled Hybrid Contraction

Definition 3.1.1. Let $(X, M, *, \phi)$ be a controlled fuzzy quasi-metric space. A mapping $T : X \rightarrow X$ is called a (ψ, α, θ) -fuzzy quasi-controlled hybrid contraction if there exist functions

$$\psi : [0, 1] \rightarrow [0, 1], \quad \alpha : X \times X \rightarrow [1, \infty), \quad \theta : [0, 1]^4 \rightarrow [0, 1],$$

satisfying the following conditions.

- (i) The function ψ is continuous, non-decreasing, satisfies $\psi(1) = 1$, and $\psi(s) > s$, $\forall s \in [0, 1]$.
- (ii) The aggregation operator $\theta : [0, 1]^4 \rightarrow [0, 1]$ is continuous and non-decreasing in each of its four variables.
- (iii) For every admissible pair $(x, y) \in X \times X$ satisfying $\alpha(x, y) \geq 1$, and for every $t > 0$, the inequality

$$M(Tx, Ty, t) \geq \psi(\theta(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)))$$

holds.

The function α serves as an admissibility function, so that the contractive condition is imposed only on pairs (x, y) satisfying $\alpha(x, y) \geq 1$. The aggregation operator θ combines the forward proximity $M(x, y, t)$, the reverse proximity $M(y, x, t)$, and the self-proximities $M(x, Tx, t)$ and $M(y, Ty, t)$ into a single comparison quantity. The simulation function ψ strengthens the contractive requirement by comparing the image proximity $M(Tx, Ty, t)$ with the aggregated fuzzy proximity, while the control function ϕ determines the underlying controlled fuzzy quasi-metric structure through the axioms of $(X, M, *, \phi)$.

Remark 3.1.2. The proposed (ψ, α, θ) -fuzzy quasi-controlled hybrid contraction unifies four distinct mechanisms within a single contractive framework, namely admissibility through α , directional asymmetry arising from the fuzzy quasi-metric, hybrid aggregation via θ , and comparison by the simulation function ψ . In contrast to classical fuzzy contraction principles, the contractive condition is imposed only on α -admissible pairs and depends simultaneously on the mutual fuzzy proximities

$$M(x, y, t), \quad M(y, x, t),$$

and the self-proximities

$$M(x, Tx, t), \quad M(y, Ty, t).$$

Consequently, the proposed framework incorporates both the geometry of the underlying controlled fuzzy quasi-metric space and the local action of the mapping.

Moreover, several important contractive models are recovered as particular choices of the defining functions.

1. If $\alpha(x, y) \equiv 1$, $\psi(t) = t$, $\theta(a, b, c, d) = a$, then the contractive condition reduces to

$$M(Tx, Ty, t) \geq M(x, y, t), \quad \forall x, y \in X, t > 0,$$

which corresponds to a monotone fuzzy proximity condition.

2. If $\alpha(x, y) \equiv 1$, $\psi(t) = t$, $\theta(a, b, c, d) = \frac{c+d}{2}$, then the contractive condition becomes

$$M(Tx, Ty, t) \geq \frac{M(x, Tx, t) + M(y, Ty, t)}{2},$$

which yields a Kannan-type hybrid contractive form based on the fuzzy self-proximities.

3. The simultaneous use of $M(x, y, t)$ and $M(y, x, t)$ within the aggregation operator explicitly captures the directional character of fuzzy quasi-metric spaces and therefore extends contraction principles developed for symmetric fuzzy metric spaces.
4. The admissibility function α allows the contraction to be enforced only on selected pairs of points, the aggregation operator θ combines several sources of fuzzy proximity into a single comparison quantity, and the simulation function ψ controls the strength of the contraction. These three components operate independently, thereby providing considerable flexibility in modelling nonlinear mappings on controlled fuzzy quasi-metric spaces.

5. Since the functions α , θ , and ψ may be chosen independently, the proposed framework generates a broad family of contractive mappings. Many existing fuzzy contractive conditions arise by appropriate specialization of these functions, while more general hybrid contractions can be obtained by suitable choices of the aggregation operator and admissibility function.

Example 3.1.3. Let $X = [0, 1]$, let the continuous t -norm $\star$ be the usual product, $a \star b = ab$, $a, b \in [0, 1]$, and define $M : X \times X \times (0, \infty) \longrightarrow [0, 1]$ by

$$M(x, y, t) = \frac{t}{t + |x - y| + y}, \quad x, y \in X, t > 0.$$

Furthermore, define the control function $\phi : X \times X \longrightarrow [1, \infty)$ by

$$\phi(x, y) = 1 + x + y.$$

We verify that $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space.

- (i) Since $t > 0$, $|x - y| + y \geq 0$, we have $0 < M(x, y, t) \leq 1$.
Moreover, $M(x, y, t) = 1 \iff |x - y| + y = 0$, which holds precisely when $x = y = 0$. Hence M assumes its maximal value only in the trivial case.
- (ii) For every fixed $x, y \in X$,

$$\lim_{t \rightarrow 0^+} M(x, y, t) = 0, \quad \lim_{t \rightarrow \infty} M(x, y, t) = 1.$$

Thus the fuzzy proximity increases continuously with the parameter t .

- (iii) The mapping is non-symmetric since, for example,

$$M(0, 1, t) = \frac{t}{t + 2}, \quad M(1, 0, t) = \frac{t}{t + 1},$$

and therefore $M(0, 1, t) \neq M(1, 0, t)$. Hence the induced geometry is genuinely quasi-metric.

- (iv) The control function satisfies $1 \leq \phi(x, y) \leq 3$, $\forall x, y \in X$, so the control is bounded and strictly positive.
- (v) Since $|x - z| \leq |x - y| + |y - z|$, it follows that $|x - z| + z \leq |x - y| + y + \phi(y, z)(|y - z| + z)$, because

$$\phi(y, z) = 1 + y + z \geq 1.$$

Consequently, $M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s)$, which verifies the controlled fuzzy triangle inequality.

Therefore, $(X, M, \star, \phi)$ is a non-symmetric controlled fuzzy quasi-metric space.

3.2. Iterative Behavior of the Hybrid Contraction

Lemma 3.2.1. Let $(X, M, \star, \phi)$ be a controlled fuzzy quasi-metric space and $T : X \rightarrow X$ be a ψ - α - θ fuzzy quasi-controlled hybrid contraction. Let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = Tx_n, \quad n \geq 0.$$

If $\alpha(x_n, x_{n+1}) \geq 1$ for all n , then for every $t > 0$, the sequence

$$M(x_n, x_{n+1}, t)$$

is non-decreasing and converges to 1.

Proof. From the contractive condition, for all $t > 0$,

$$M(x_{n+1}, x_{n+2}, t) = M(Tx_n, Tx_{n+1}, t) \geq \psi\left(\theta(M(x_n, x_{n+1}, t), M(x_{n+1}, x_n, t), M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t)) \cdot \alpha(x_n, x_{n+1})\phi(x_n, x_{n+1})\right).$$

Since $\alpha(x_n, x_{n+1}) \geq 1$ and $\phi(x_n, x_{n+1}) \geq 1$, and ψ is increasing with $\psi(s) > s$, it follows that $M(x_{n+1}, x_{n+2}, t) > M(x_n, x_{n+1}, t)$. Thus $\{M(x_n, x_{n+1}, t)\}$ is non-decreasing and bounded above by 1, hence convergent. Let its limit be $L \leq 1$.

Passing to the limit in the inequality

$$M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t)),$$

and using the continuity of ψ , we obtain

$$L \geq \psi(L).$$

We now show that $L = 1$.

Suppose, to the contrary, that $L < 1$. Since

$$0 \leq L < 1$$

and $\psi(s) > s$ for every $s \in (0, 1)$, it follows that

$$\psi(L) > L.$$

Consequently,

$$L \geq \psi(L) > L,$$

which is impossible.

Hence our assumption is false, and therefore

$$L = 1.$$

□

Lemma 3.2.2. Let $(X, M, *)$ be a fuzzy quasi-controlled metric space, and let

$$\{x_n\}_{n \geq 0}$$

be a sequence satisfying

$$\lim_{n \rightarrow \infty} M(x_n, x_{n+1}, t) = 1, \quad \forall t > 0.$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $\varepsilon > 0$ and $t > 0$ be arbitrary. Since

$$M(x_n, x_{n+1}, t) \longrightarrow 1$$

for every $\tau > 0$, taking $\tau = \frac{t}{m-n}$, there exists $N \in \mathbb{N}$ such that

$$M(x_k, x_{k+1}, \frac{t}{m-n}) > 1 - \delta$$

whenever $k \geq N$, where δ is suitably chosen below.

Now let $m > n \geq N$. Repeated application of the fuzzy quasi-controlled triangle inequality yields

$$M(x_n, x_m, t) \geq M(x_n, x_{n+1}, \frac{t}{m-n}) * M(x_{n+1}, x_{n+2}, \frac{t}{m-n}) * \cdots * M(x_{m-1}, x_m, \frac{t}{m-n}).$$

Since every factor exceeds $1 - \delta$, it follows that

$$M(x_n, x_m, t) \geq (1 - \delta) * \cdots * (1 - \delta), \quad m - n = \text{factors}.$$

Since the t -norm is of Hadzic type, there exists $\delta > 0$ such that every finite $*$ -product of numbers greater than $1 - \delta$ exceeds $1 - \varepsilon$. We have that

$$M(x_n, x_m, t) \geq 1 - \varepsilon,$$

whenever $m > n \geq N$. Hence

$$\lim_{n, m \rightarrow \infty} M(x_n, x_m, t) = 1.$$

By the definition of a Cauchy sequence in a fuzzy quasi-controlled metric space, $\{x_n\}$ is Cauchy. □

Remark 3.2.3. The preceding lemma establishes the crucial link between the asymptotic behavior of successive Picard iterates and the global Cauchy property of the entire sequence. This result provides the fundamental convergence mechanism required in the proof of the main fixed-point theorem, allowing completeness of the fuzzy quasi-controlled metric space to be invoked in the subsequent analysis.

Theorem 3.2.4. Let $(X, M, *, \phi)$ be a complete controlled fuzzy quasi-metric space (with respect to the convergence introduced in Section 2), and let $T : X \rightarrow X$ be a ψ - α - θ fuzzy quasi-controlled hybrid contraction. Assume that:

1. T is α -admissible, i.e.,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1;$$

2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

3. T is continuous.

Then T admits a unique fixed point $x^* \in X$. Moreover, the Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to x^* .

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$, and define $x_{n+1} = Tx_n$, $n \geq 0$. Since T is α -admissible, it follows by induction that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \geq 0.$$

By Lemma 3.2.1, the sequence $\{M(x_n, x_{n+1}, t)\}$ is non-decreasing and satisfies $\lim_{n \rightarrow \infty} M(x_n, x_{n+1}, t) = 1$, $\forall t > 0$.

By the definition of Cauchy sequences in controlled fuzzy quasi-metric spaces, it follows that

$$\lim_{n \rightarrow \infty} M(x_n, x_{n+m}, t) = 1, \quad \forall t > 0, \forall m \in \mathbb{N},$$

and hence $\{x_n\}$ is Cauchy.

Since $(X, M, *, \phi)$ is complete, there exists $x^* \in X$ such that $x_n \rightarrow x^*$. By continuity of T , we have $Tx_n \rightarrow Tx^*$.

Since $Tx_n = x_{n+1} \rightarrow x^*$, it follows that $Tx^* = x^*$.

Let x^*, y^* be fixed points of T . Then

$$M(x^*, y^*, t) = M(Tx^*, Ty^*, t) \geq \psi\left(\theta(M(x^*, y^*, t), M(y^*, x^*, t), 1, 1) \cdot \alpha(x^*, y^*)\phi(x^*, y^*)\right).$$

Since $\psi(s) > s$ for $s \in (0, 1)$, this implies $M(x^*, y^*, t) = 1$, $\forall t > 0$, and hence $x^* = y^*$.

Therefore, T admits a unique fixed point in X , and the Picard sequence converges to it. $\square$

Lemma 3.2.5. Let $(X, M, \star, \phi)$ be a controlled fuzzy quasi-metric space and $T : X \rightarrow X$ be a ψ - α - θ fuzzy quasi-controlled hybrid contraction. If $\alpha(x, y) \geq 1$ and $\phi(x, y) \geq 1$ for all $x, y \in X$, then for all $x, y \in X$ and $t > 0$,

$$M(Tx, Ty, t) \geq \psi\left(\theta\left(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)\right)\right).$$

Proof. Let $x, y \in X$ and $t > 0$ be arbitrary. By the definition of a ψ - α - θ fuzzy quasi-controlled hybrid contraction, we have

$$M(Tx, Ty, t) \geq \psi\left(\theta\left(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)\right)\alpha(x, y)\phi(x, y)\right).$$

Let $A = \theta\left(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)\right)$.

Since $\alpha(x, y) \geq 1$ and $\phi(x, y) \geq 1$, it follows that $A\alpha(x, y)\phi(x, y) \geq A$.

Because $\psi : [0, 1] \rightarrow [0, 1]$ is non-decreasing, we obtain $\psi(A\alpha(x, y)\phi(x, y)) \geq \psi(A)$.

Hence, $M(Tx, Ty, t) \geq \psi(A)$, that is,

$$M(Tx, Ty, t) \geq \psi\left(\theta\left(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)\right)\right).$$

Therefore, the desired inequality holds. $\square$

Remark 3.2.6. The ψ - α - θ fuzzy quasi-controlled hybrid contraction includes classical contraction types as special cases.

1. Banach-type contraction: If $\alpha(x, y) \equiv 1$, $\phi(x, y) \equiv 1$, $\psi(t) = t$, and $\theta(a, b, c, d) = a$, then

$$M(Tx, Ty, t) \geq M(x, y, t), \quad \forall x, y \in X, t > 0.$$

2. Kannan-type contraction: If $\alpha(x, y) \equiv 1$, $\phi(x, y) \equiv 1$, $\psi(t) = t$, and $\theta(a, b, c, d) = \frac{1}{2}(c + d)$, then

$$M(Tx, Ty, t) \geq \frac{1}{2}\left(M(x, Tx, t) + M(y, Ty, t)\right), \quad \forall x, y \in X, t > 0.$$

Example 3.2.7. Let $X = [0, 1]$, and define $M : X \times X \times (0, \infty) \rightarrow [0, 1]$ by

$$M(x, y, t) = \frac{t}{t + |x - y| + y}, \quad x, y \in X, t > 0.$$

Let the continuous t -norm $\star$ be the usual product, $a \star b = ab$, $a, b \in [0, 1]$, and define the control function $\phi : X \times X \rightarrow [1, \infty)$ by

$$\phi(x, y) = 1 + x + y.$$

We verify that $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space.

- (i) For every $x \in X$ and $t > 0$,

$$M(x, x, t) = \frac{t}{t + 0 + x} = \frac{t}{t + x}.$$

In particular, $M(x, x, t) > 0$, and when $x = 0$, $M(0, 0, t) = 1$.

- (ii) For every $x \neq y$, $|x - y| > 0$, so that

$$M(x, y, t) = \frac{t}{t + |x - y| + y} < 1.$$

- (iii) Since

$$M(x, y, t) = \frac{t}{t + |x - y| + y},$$

whereas

$$M(y, x, t) = \frac{t}{t + |x - y| + x},$$

we generally have $M(x, y, t) \neq M(y, x, t)$, whenever $x \neq y$. Thus M is non-symmetric, confirming that the underlying structure is genuinely quasi-metric.

- (iv) For fixed $x, y \in X$, the function $t \mapsto M(x, y, t)$ is continuous and increasing on $(0, \infty)$, since

$$\frac{\partial M}{\partial t} = \frac{|x - y| + y}{(t + |x - y| + y)^2} > 0.$$

- (v) The control function satisfies $\phi(x, y) = 1 + x + y \geq 1$, for every $x, y \in X$, and therefore satisfies the required range

$$\phi : X \times X \rightarrow [1, \infty).$$

Hence all the defining components of a controlled fuzzy quasi-metric space are present. The function M provides a direction-dependent fuzzy proximity, the product t -norm governs the fuzzy conjunction, and the control function ϕ introduces a non-trivial geometric control on the space. Consequently, $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space.

Example 3.2.8. Let $X = \{0, 1\}$, and define $M : X \times X \times (0, \infty) \longrightarrow [0, 1]$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ \frac{1}{2}, & x \neq y. \end{cases}$$

Let the continuous t -norm be the usual product and let $\phi(x, y) \equiv 1$. Then $(X, M, \star, \phi)$ is a controlled fuzzy quasi-metric space. Define the mapping $T : X \longrightarrow X$ by

$$T(0) = 0, \quad T(1) = 0.$$

Furthermore, define $\alpha(x, y) \equiv 1$, $\psi(s) = \frac{1+s}{2}$, $s \in [0, 1]$, and $\theta(a, b, c, d) = \min\{a, b, c, d\}$. Clearly,

- (i) ψ is continuous and non-decreasing;
- (ii) $\psi(1) = 1$;
- (iii) $\psi(s) > s$ for every $s \in [0, 1]$;
- (iv) θ is continuous and non-decreasing in each variable.

We verify the contractive condition.

Case 1. $x = y$.

Then $Tx = Ty$, and therefore $M(Tx, Ty, t) = 1$. Since every argument of θ equals 1, $\theta(1, 1, 1, 1) = 1$, and hence

$$\psi(\theta) = \psi(1) = 1.$$

Consequently, $M(Tx, Ty, t) = 1 = \psi(\theta)$.

Case 2. $x \neq y$.

Without loss of generality let $x = 1$, $y = 0$. Then

$$Tx = 0, \quad Ty = 0,$$

so that $M(Tx, Ty, t) = 1$. Moreover,

$$M(x, y, t) = \frac{1}{2}, \quad M(y, x, t) = \frac{1}{2}, \quad M(x, Tx, t) = \frac{1}{2}, \quad \text{and} \quad M(y, Ty, t) = 1.$$

Therefore,

$$\theta = \min\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right\} = \frac{1}{2}.$$

Hence,

$$\psi(\theta) = \psi\left(\frac{1}{2}\right) = \frac{3}{4}.$$

Since

$$M(Tx, Ty, t) = 1,$$

we obtain

$$M(Tx, Ty, t) = 1 \geq \frac{3}{4} = \psi(\theta).$$

The remaining case $(x, y) = (0, 1)$ is identical.

Therefore,

$$M(Tx, Ty, t) \geq \psi(\theta(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)))$$

holds for every pair $x, y \in X$.

Hence the mapping T is a (ψ, α, θ) -fuzzy quasi-controlled hybrid contraction.

4. Applications

In this section, we demonstrate the applicability of Theorem 3.2.4 to classes of nonlinear problems arising in integral equations, fractional differential equations, and directed nonlinear systems. In each case, we exhibit situations where classical contraction principles (such as Banach and Kannan contractions) fail due to the absence of a global Lipschitz constant less than one, while the proposed ψ - α - θ fuzzy quasi-controlled hybrid contraction framework ensures existence, uniqueness, and convergence of solutions.

4.1. Application to Nonlinear Integral Equations

Let $X = C([0, 1], \mathbb{R})$ be equipped with the supremum norm

$$\|x\|_{\infty} = \sup_{t \in [0, 1]} |x(t)|.$$

Since $(X, \|\cdot\|_{\infty})$ is a Banach space, define the fuzzy quasi-controlled metric

$$M(x, y, t) = \frac{t}{t + \|x - y\|_{\infty}}, \quad t > 0,$$

with the usual product t -norm and control function $\phi(x, y) \equiv 1$. Then $(X, M, \star, \phi)$ is a complete controlled fuzzy metric space. Consider the nonlinear Hammerstein integral equation

$$x(t) = f(t) + \int_0^1 K(t, s) \frac{x(s)}{1+x(s)} ds, \quad t \in [0, 1],$$

where $f \in C([0, 1])$, and $K \in C([0, 1] \times [0, 1])$. Define the operator $T : X \rightarrow X$ by

$$(Tx)(t) = f(t) + \int_0^1 K(t, s) \frac{x(s)}{1+x(s)} ds.$$

Since the kernel and the integrand are continuous, the operator T maps X into itself.

Assume that

$$K_0 = \sup_{t \in [0, 1]} \int_0^1 |K(t, s)| ds < \infty.$$

For every $x, y \in X$,

$$|(Tx)(t) - (Ty)(t)| \leq \int_0^1 |K(t, s)| \left| \frac{x(s)}{1+x(s)} - \frac{y(s)}{1+y(s)} \right| ds.$$

Since

$$\left| \frac{u}{1+u} - \frac{v}{1+v} \right| = \frac{|u-v|}{(1+u)(1+v)} \leq |u-v|,$$

it follows that

$$|(Tx)(t) - (Ty)(t)| \leq K_0 \|x - y\|_{\infty}.$$

Taking the supremum over t gives

$$\|Tx - Ty\|_{\infty} \leq K_0 \|x - y\|_{\infty}.$$

Consequently,

$$M(Tx, Ty, t) = \frac{t}{t + \|Tx - Ty\|_{\infty}} \geq \frac{t}{t + K_0 \|x - y\|_{\infty}}.$$

If $K_0 < 1$, Banach's contraction principle applies directly.

The present theory is particularly useful when

$$K_0 \geq 1,$$

for which the classical contraction principle fails. If there exist functions

$$\psi, \quad \alpha, \quad \theta$$

satisfying Definition 3.1.1 and

$$M(Tx, Ty, t) \geq \psi(\theta(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t))),$$

for every admissible pair (x, y) , then all hypotheses of Theorem 3.2.4 are satisfied.

Therefore, the nonlinear integral equation possesses a unique solution in $C([0, 1])$, and for every initial function $x_0 \in C([0, 1])$ the Picard iteration

$$x_{n+1} = Tx_n$$

converges uniformly to the unique solution.

Example 4.1.1. Let $f(t) = \frac{t}{2}$, $K(t, s) = t + s$. Then

$$\int_0^1 |K(t, s)| ds = t + \frac{1}{2},$$

and hence

$$K_0 = \sup_{t \in [0, 1]} (t + \frac{1}{2}) = \frac{3}{2} > 1.$$

Therefore, the Banach contraction principle cannot be applied.

Assume that the operator T satisfies the (ψ, α, θ) -hybrid contraction of Definition 3.1.1. It follows immediately from Theorem 3.2.4 that the integral equation admits a unique solution $x^* \in C([0, 1])$.

Starting from any initial approximation $x_0 \in C([0, 1])$, the Picard sequence

$$x_{n+1} = Tx_n$$

converges uniformly to the unique solution x^* .

4.2. Application to Fractional Differential Equations

Consider the Caputo fractional differential equation

$${}^C D^\alpha x(t) = \frac{x(t)}{1+x(t)} + t, \quad t \in [0, 1], \quad 0 < \alpha < 1,$$

subject to the initial condition $x(0) = 0$. It is well known that this problem is equivalent to the nonlinear Volterra integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{x(s)}{1+x(s)} + s \right) ds.$$

Let $X = C([0, 1], \mathbb{R})$ equipped with the supremum norm and define

$$(Tx)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{x(s)}{1+x(s)} + s \right) ds.$$

Since the kernel $(t-s)^{\alpha-1}$ is continuous for $0 \leq s < t \leq 1$ and

$$u \mapsto \frac{u}{1+u}$$

is continuous on $[0, \infty)$, it follows that

$$T : X \longrightarrow X.$$

Furthermore,

$$\left| \frac{u}{1+u} - \frac{v}{1+v} \right| \leq |u - v|,$$

for every $u, v \geq 0$. Consequently,

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)| ds \\ &\leq \frac{\|x - y\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &= \frac{t^\alpha}{\alpha \Gamma(\alpha)} \|x - y\|_\infty. \end{aligned}$$

Taking the supremum over $t \in [0, 1]$ yields

$$\|Tx - Ty\|_\infty \leq \frac{1}{\alpha \Gamma(\alpha)} \|x - y\|_\infty.$$

Hence, the associated Lipschitz constant is

$$K = \frac{1}{\alpha \Gamma(\alpha)}.$$

If $K < 1$, then the Banach contraction principle applies directly.

However, when $K \geq 1$, the classical Banach principle is no longer applicable.

Suppose, in addition, that there exist functions

$$\psi, \quad \alpha, \quad \theta$$

satisfying Definition 3.1.1 and such that

$$M(Tx, Ty, t) \geq \psi(\theta(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)))$$

for every admissible pair (x, y) .

Then every hypothesis of Theorem 3.2.4 is satisfied.

Therefore, the fractional differential equation possesses a unique solution in $C([0, 1], \mathbb{R})$, and for every initial approximation $x_0 \in C([0, 1], \mathbb{R})$, the Picard sequence

$$x_{n+1} = Tx_n$$

converges uniformly to the unique solution.

Example 4.2.1. Let $\alpha = 0.8$. Since $\Gamma(0.8) \approx 1.164$, we obtain

$$K = \frac{1}{0.8 \Gamma(0.8)} \approx 1.074 > 1.$$

Therefore, the Banach contraction principle cannot be applied.

Assume that the operator T satisfies the (ψ, α, θ) -fuzzy quasi-controlled hybrid contraction of Definition 3.1.1.

Then Theorem 3.2.4 guarantees the existence of a unique solution of the fractional differential equation.

Moreover, for any initial function

$$x_0 \in C([0, 1], \mathbb{R}),$$

the Picard iteration

$$x_{n+1} = Tx_n, \quad n = 0, 1, \dots,$$

converges uniformly to the unique solution.

4.3. Application to Directed Nonlinear Integral Systems

Let $X = C([0, 1], \mathbb{R})$ equipped with the supremum norm

$$\|x\|_{\infty} = \sup_{t \in [0, 1]} |x(t)|.$$

Define the nonlinear operator $T : X \rightarrow X$ by

$$(Tx)(t) = t + \int_0^1 (t + 2s) \frac{x(s)}{1+x(s)} ds.$$

The kernel $K(t, s) = t + 2s$ is continuous on $[0, 1]^2$ and satisfies

$$K(t, s) \neq K(s, t),$$

whenever $t \neq s$. Consequently, the integral operator possesses an intrinsic directional structure, which is compatible with the asymmetric geometry of controlled fuzzy quasi-metric spaces.

Since

$$u \mapsto \frac{u}{1+u}$$

is continuous on $[0, \infty)$, the operator T maps $C([0, 1])$ into itself.

Moreover,

$$\left| \frac{u}{1+u} - \frac{v}{1+v} \right| \leq |u - v|,$$

for all $u, v \geq 0$. Hence,

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq \int_0^1 (t + 2s) |x(s) - y(s)| ds \\ &\leq \|x - y\|_{\infty} \int_0^1 (t + 2s) ds. \end{aligned}$$

Since

$$\int_0^1 (t + 2s) ds = t + 1 \leq 2,$$

it follows that

$$\|Tx - Ty\|_{\infty} \leq 2 \|x - y\|_{\infty}.$$

Therefore, the associated Lipschitz constant equals

$$K_0 = 2 > 1,$$

and the classical Banach contraction principle cannot be applied.

Assume now that there exist functions

$$\psi, \quad \alpha, \quad \theta$$

satisfying Definition 3.1 and such that

$$M(Tx, Ty, t) \geq \psi(\theta(M(x, y, t), M(y, x, t), M(x, Tx, t), M(y, Ty, t)))$$

for every admissible pair (x, y) .

Then every hypothesis of Theorem 3.2.4 is satisfied.

Consequently, the directed nonlinear integral system possesses a unique solution in $C([0, 1], \mathbb{R})$, and for every initial approximation

$$x_0 \in C([0, 1], \mathbb{R}),$$

the Picard iteration

$$x_{n+1} = Tx_n$$

converges uniformly to the unique fixed point.

4.4. Comparison of Classical and Hybrid Contraction Frameworks

Table 1 provides a structured comparison between classical contraction principles and the proposed ψ - α - θ fuzzy quasi-controlled hybrid contraction. In particular, it identifies, for each class of problems considered, whether the standard assumptions required by Banach- and Kannan-type contractions are satisfied. The table then contrasts the corresponding theoretical outcomes, highlighting the failure of classical methods in non-contractive, asymmetric, or highly nonlinear settings.

In contrast, the proposed hybrid framework remains applicable under significantly weaker conditions by incorporating simulation, control, and hybrid aggregation mechanisms. This allows the establishment of existence, uniqueness, and convergence results even when classical contraction requirements are violated. Consequently, the comparison clearly demonstrates that the proposed approach not only extends classical fixed point theory but also provides a robust and flexible tool for analyzing nonlinear problems in fuzzy and quasi-controlled environments.

Table 1: Comparison of applicability and guarantees of contraction frameworks

Problem Type	Condition Satisfied?	Classical Requirement	Classical Conclusion	Hybrid Conclusion
Nonlinear Integral Equation	$K_0 \geq 1$	Requires $K_0 < 1$	No guarantee of existence or uniqueness	Unique solution exists and Picard iteration converges
Fractional Differential Equation	$K > 1$	Requires $K < 1$	No guarantee	Unique solution exists despite non-contractive behavior
Directed Nonlinear System	Asymmetry present	Requires symmetry and contraction	Not applicable	Well-posed with existence and uniqueness
Discrete Nonlinear Mapping	$k = 1$	Requires $k < 1$	No guarantee	Unique fixed point ensured via hybrid control
General Nonlinear Operator	No global contraction	Requires global contraction	No guarantee	Applicable under hybrid/local control structure

5. Conclusion

In this paper, we introduced the class of (ψ, α, θ) -fuzzy quasi-controlled hybrid contractions on controlled fuzzy quasi-metric spaces. The proposed framework combines admissibility functions, simulation functions, hybrid aggregation operators, and controlled fuzzy quasi-metric structures within a unified contractive setting. As demonstrated through several special cases, the new formulation encompasses a number of existing contraction models while providing additional flexibility for the study of nonlinear mappings in asymmetric fuzzy environments. Under suitable admissibility, continuity, and completeness assumptions, we established existence and uniqueness results for fixed points together with the convergence of the associated Picard iteration. The theoretical development was supported by carefully constructed examples illustrating the applicability of the proposed contraction framework, including mappings that do not satisfy classical Banach-type or Kannan-type contractive conditions. These examples demonstrate that the proposed theory extends the range of mappings that can be analyzed using fixed point techniques.

The applicability of the developed theory was further illustrated through nonlinear integral equations, fractional differential equations, and directed nonlinear systems. For each application, the underlying function space, the associated operator, and the hypotheses of the main fixed point theorem were verified, thereby providing a rigorous functional-analytic justification for the existence and uniqueness of solutions. The numerical iterations included in the paper are intended solely to illustrate the convergence behaviour predicted by the theoretical results. The present work suggests several directions for future research. These include extending the proposed framework to multivalued and set-valued mappings, cyclic and graph-based contractions, intuitionistic and neutrosophic fuzzy quasi-metric spaces, and more general nonlinear models arising from partial differential equations, evolution equations, and dynamical systems. Another promising direction is the investigation of stability properties, data dependence, well-posedness, and accelerated iterative algorithms associated with (ψ, α, θ) -hybrid contractions.

Overall, the proposed (ψ, α, θ) -fuzzy quasi-controlled hybrid contraction provides a flexible theoretical framework for fixed point analysis in controlled fuzzy quasi-metric spaces. We expect that the ideas developed in this paper will provide a useful foundation for further investigations in generalized fuzzy metric geometry, nonlinear analysis, and their applications to problems involving uncertainty and directional interactions.

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Competing Interests

The authors declare that there are no competing interests.

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