

Modified Adomian Decomposition Method for Third -Order Emden-Fowler Equation

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Abstract

In this study, we investigate third-order Emden-Fowler equations using the modified Adomian Decomposition Method (MADM). Six operators tailored to this framework are introduced and applied to analyze the equations under specific initial and boundary conditions. The results demonstrate that the solutions obtained via the modified Adomian Decomposition Method converge effectively to the exact solutions of the third-order Emden-Fowler equations. This work not only enhances the understanding of these complex equations but also highlights the effectiveness of the MADM in producing accurate solutions under varying conditions.

Keywords: Third -Order Emden-Fowler Equation, Modified Adomian Decomposition Method, Initial and boundary Conditions, Convergence Analysis.

1. Introduction

Third-order Emden-Fowler equations are a class of differential equations that arise in various fields, particularly in astrophysics and fluid dynamics [1]. They represent a generalization of the well-known Lane-Emden equations, extending the concept to higher-order derivatives. This equation typically takes the form:

$$y''' + \frac{c}{x}y'' + N(y) = g(x), \quad (1)$$

where c is a real number, and $g(x)$ is a function. The third order signifies that these equations involve the third derivative of the unknown function y with respect to the independent variable x . The study of third-order Emden-Fowler equations includes the analysis of the existence and uniqueness of solutions, as well as their asymptotic behavior. These equations often present significant challenges due to their nonlinear nature, which motivates the use of various numerical methods for obtaining effective solutions. Important applications of such equations arise in models of stellar structure, where they are used describe the behavior of polytropic stars, as well as in other nonlinear phenomena in physics and engineering [2–6]. The Adomian Decomposition Method (ADM) is a semi-analytical technique widely used to solve ordinary and partial nonlinear differential equations. Developed by George Adomian primarily in the late 20th century, the method decomposes a complex problem into simpler problems that can be solved iteratively. This approach allows for the construction of a series solution whose components are computed recursively. ADM has demonstrated effectiveness in various fields, including applied mathematics, engineering, and physics [7–9]. Several foundational studies have investigated the theoretical aspects and practical applications of ADM, particularly its robustness for initial-value problems (IVPs) and boundary-value problems (BVPs). To improve convergence properties and extend applicability to more complex nonlinear systems, numerous modifications of the basic ADM have been proposed [10–18].

The numerical solution of third-order Emden-Fowler equations has also motivated the development and combination of various computational techniques aimed at enhancing accuracy and efficiency. Several studies have employed numerical methods such as the Variational Iteration Method (VIM) [1, 19], as well as ADM and quartic trigonometric B-spline method [5] to solve third-order Emden-Fowler equations. The aim of this article is to investigate and enhance the analysis of third-order Emden-Fowler equations using the MADM. Specifically, the study seeks to introduce a tailored framework comprising six specialized operators to facilitate effective solutions under various initial and boundary conditions.

ADM often suffers from slow convergence near singular points. The VIM requires the accurate identification of Lagrange multipliers, which becomes complex for higher-order problems. Furthermore, numerical approaches such as Artificial Neural Network (ANN) techniques

involve high computational costs and complex training procedures. The proposed MADM in this study addresses these issues by providing flexible operator selection and rapid convergence to the exact solution. This work is an original research article focusing on the development and application of six new modified operators specifically designed for third-order singular problems.

2. New Modified ADM

In this section, we introduce the six modified operators that form the core of our proposed framework for MADM. Each operator has been specifically tailored to enhance the solution process for the third-order Emden-Fowler equations. Consider Eq.(1), in an operator form, Eq.(1) becomes

$$Ly = g(x) - N(y), \quad (2)$$

Ly take the following form.

$$L_1(y) = x^{-c} \frac{d}{dx} x^c \frac{d^2}{dx^2} (y),$$

$$L_2(y) = x^{-c} \frac{d}{dx} x^{c-1} \frac{d}{dx} x^2 \frac{d}{dx} x^{-1} (y),$$

$$L_3(y) = x^{-2} \frac{d}{dx} x^{3-c} \frac{d}{dx} x^c \frac{d}{dx} x^{-1} (y), \quad (3)$$

$$L_4(y) = x^{-1} \frac{d}{dx} x^{2-c} \frac{d}{dx} x^{c-1} \frac{d}{dx} (y),$$

$$L_5(y) = x^{-1} \frac{d^2}{dx^2} x^{3-c} \frac{d}{dx} x^{c-2} (y),$$

$$L_6(y) = x^{-2} \frac{d}{dx} x^2 \frac{d}{dx} x^{2-c} \frac{d}{dx} x^{c-2} (y),$$

The inverse of above operators as follows

$$L_1^{-1} = \int_0^x \int_0^x x^{-c} \int_0^x x^c (y) dx dx dx,$$

$$L_2^{-1} (y) = x \int_a^x x^{-2} \int_0^x x^{1-c} \int_0^x x^c (y) dx dx dx,$$

$$L_3^{-1} (y) = x \int_a^x x^{-c} \int_b^x x^{c-3} \int_0^x x^2 (y) dx dx dx, \quad (4)$$

$$L_4^{-1} (y) = \int_0^x x^{1-c} \int_a^x x^{c-2} \int_0^x x (y) dx dx dx,$$

$$L_5^{-1} (y) = x^{2-c} \int_a^x x^{c-3} \int_0^x \int_0^x x (y) dx dx dx,$$

$$L_6^{-1} (y) = x^{2-c} \int_a^x x^{c-2} \int_b^x x^{-2} \int_0^x x^2 (y) dx dx dx.$$

It should be noted that the introduction of multiple modified linear operators in this study is intended to highlight the flexibility and robustness of the proposed MADM framework rather than to produce distinct solutions in all cases. Depending on the mathematical structure of the third-order Emden-Fowler equation, different operators may reduce to equivalent forms, leading to identical series solutions. This observation reflects the consistency of the method and does not diminish its novelty or effectiveness.

To get y , take L^{-1} to both sides of Eq.(3).

Then

$$y = \Omega + L^{-1}g(x) - L^{-1}N(y), \quad (5)$$

where Ω arising from auxiliary conditions.

The solution in ADM decomposition to [9]

$$y_0 = \Omega + L^{-1}g(x),$$

$$y_{n+1} = -L^{-1}N(y),$$

where

$$N(y) = \sum_{n=0}^{\infty} A_n,$$

A_n called Adomian polynomial and formed by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i\right) \right], \text{ where } \lambda = 0, n = 0, 1, 2, 3, \dots \tag{6}$$

That is

$$y_{n+1} = -L^{-1}A_n,$$

then

$$y_1 = -L^{-1}A_0,$$

$$y_2 = -L^{-1}A_1,$$

$$y_3 = -L^{-1}A_2,$$

and so on.

Based on the above, the solution by the ADM is given as series

$$y(x) = y_0 + y_1 + y_2 + y_3 + \dots = \sum_{n=0}^{\infty} y_n. \tag{7}$$

3. Convergence Analysis

In this section, we provide a theoretical justification for the convergence of the proposed Modified Adomian Decomposition Method (MADM) applied to the third-order Emden-Fowler equation. The convergence is established using the Banach Fixed Point Theorem.

Theorem 1. Let $B = (C[J], \|\cdot\|)$ be a Banach space of all continuous functions on the interval $J = [0, T]$ with the supremum norm $\|y\| = \max_{x \in J} |y(x)|$. The series solution $u(x) = \sum_{n=0}^{\infty} y_n(x)$ generated by the modified operator L_i converges to a unique solution in B if there exists a constant $0 < \gamma < 1$ such that $\|y_{n+1}\| \leq \gamma \|y_n\|$ for all $n \in \mathbb{N}$.

Proof. Consider the general form of the third-order Emden-Fowler equation:

$$L_i y(x) + N y(x) = g(x),$$

where L_i denotes the i -th modified linear operator and N represents the nonlinear term. The MADM recurrence relation is defined as:

$$y_0 = \Omega(x),$$

$$y_{n+1} = -L_i^{-1}[N(y_n)], \quad n \geq 0.$$

The modified inverse operator L_1^{-1} is defined based on the structure provided in Section 2 as follows:

$$L_1^{-1}[\cdot] = \int_0^x \int_0^x x^{-c} \int_0^x x^c (\cdot) dx dx dx.$$

Assuming the nonlinear term $N(y)$ satisfies the Lipschitz condition $\|N(y) - N(z)\| \leq M \|y - z\|$ for a positive constant M . We define the convergence parameter γ by evaluating the operator norm:

$$\gamma = \max_{x \in J} \left| \int_0^x \int_0^x x^{-c} \int_0^x x^c M dx dx dx \right|.$$

By calculating the nested integrals, the value of γ for the operator L_1 is found to be:

$$\gamma = \frac{MT^3}{6(-1+c)}.$$

For any $k > m$, the partial sums $S_k = \sum_{n=0}^k y_n$ satisfy:

$$\|S_k - S_m\| = \left\| \sum_{n=m+1}^k y_n \right\| \leq \sum_{n=m+1}^k \|y_n\| \leq \sum_{n=m+1}^k \gamma^n \|y_0\|.$$

Which simplifies to:

$$\|S_k - S_m\| \leq \frac{\gamma^{m+1}(1 - \gamma^{k-m})}{1 - \gamma} \|y_0\|.$$

Since $0 < \gamma < 1$, as $m \rightarrow \infty$, it follows that $\gamma^{m+1} \rightarrow 0$. Thus, $\|S_k - S_m\| \rightarrow 0$, proving that $\{S_k\}$ is a Cauchy sequence in the Banach space B . By the completeness of B , the series $\sum_{n=0}^{\infty} y_n(x)$ converges to a unique solution $y(x)$. \square

Remark 2. The same logical framework applies to all six modified operators (L_1 to L_6). While the specific integral kernel of each L_i^{-1} varies, the parameters c are strategically chosen to ensure that $\gamma < 1$, guaranteeing rapid convergence as demonstrated in the numerical examples.

4. Numerical Application

In this section, we examine three distinct examples, encompassing both linear and nonlinear cases. Utilizing the six specialized operators introduced in our framework, we provide detailed explanations and analyses for each example. This approach allows us to effectively demonstrate the applicability and efficacy of the MADM in addressing the nuances of these equations. Through this investigation, we aim to highlight the strengths of our method in solving a diverse range of problems within the context of third-order Emden-Fowler equations.

Example 1. Consider the third-order Emden-Fowler type [20]

$$y''' + \frac{4}{x}y'' = (x^6 + 10x^3 + 10)y(x), \quad 0 \leq x \leq 1. \quad (8)$$

With initial and boundary conditions

$$y(0) = 1, y'(0) = 0, y''(0) = 0,$$

$$y(0) = 1, y(0.2) = 1.00267, y'(0) = 0,$$

$$y(0) = 1, y(0.1) = 1.00033, y'(0.1) = 0.100033,$$

Eq.(8) in operator form

$$Ly = (x^6 + 10x^3 + 10)y(x), \quad (9)$$

if we put $c = 4$ in Eq.(3) we get the following six operators

$$L_1(y) = x^{-4} \frac{d}{dx} x^4 \frac{d^2}{dx^2} (y),$$

$$L_2(y) = x^{-4} \frac{d}{dx} x^3 \frac{d}{dx} x^2 \frac{d}{dx} x^{-1} (y),$$

$$L_3(y) = x^{-2} \frac{d}{dx} x^{-1} \frac{d}{dx} x^4 \frac{d}{dx} x^{-1} (y),$$

$$L_4(y) = x^{-1} \frac{d}{dx} x^{-2} \frac{d}{dx} x^3 \frac{d}{dx} (y),$$

$$L_5(y) = x^{-1} \frac{d^2}{dx^2} x^{-1} \frac{d}{dx} x^2 (y),$$

$$L_6(y) = x^{-2} \frac{d}{dx} x^2 \frac{d}{dx} x^{-2} \frac{d}{dx} x^2 (y),$$

and the six inverse operators

$$L_1^{-1} = \int_0^x \int_0^x x^{-4} \int_0^x x^4 (y) dx dx dx,$$

$$L_2^{-1} (y) = x \int_{0.2}^x x^{-2} \int_0^x x^{-3} \int_0^x x^4 (y) dx dx dx,$$

$$L_3^{-1} (y) = x \int_0^x x^{-4} \int_0^x x \int_0^x x^2 (y) dx dx dx,$$

$$L_4^{-1} (y) = \int_0^x x^{-3} \int_0^x x^2 \int_0^x x (y) dx dx dx,$$

$$L_5^{-1} (y) = x^{-2} \int_0^x x \int_{0.2}^x \int_0^x x (y) dx dx dx,$$

$$L_6^{-1} (y) = x^{-2} \int_0^x x^2 \int_0^x x^{-2} \int_0^x x^2 (y) dx dx dx.$$

Taking L^{-1} on both sides of Eq.(9) and applying the steps of section 2, we get the following result:

The result by L_1

$$\begin{aligned} y_0 &= 1, \\ y_1 &= 0.333333x^3 + 0.0416667x^6 + 0.00126263x^9, \\ y_2 &= 0.0138889x^6 + 0.00473485x^9 + 0.000412677x^{12} + 0.0000152081x^{15} + 2.06311 \cdot 10^{-7}x^{18}, \\ y_3 &= 0.000175365x^9 + 0.000100778x^{12} + 0.0000183093x^{15} + 1.47283 \cdot 10^{-6}x^{18} + 5.86771 \cdot 10^{-8}x^{21} + 1.2034 \cdot 10^{-9}x^{24} + \\ &\quad 1.01342 \cdot 10^{-11}x^{27}, \end{aligned}$$

the series solution by L_1 is

$$\begin{aligned} y &= y_0 + y_1 + y_2 + y_3 = \\ &1. + 0.333333x^3 + 0.0555556x^6 + 0.00617284x^9 + 0.000513454x^{12} + 0.0000335174x^{15} + 1.67914 \cdot 10^{-6}x^{18} + 5.86771 \cdot 10^{-8}x^{21} + \\ &\quad 1.2034 \cdot 10^{-9}x^{24} + 1.01342 \cdot 10^{-11}x^{27}. \end{aligned}$$

The result by L_2

$$y_0 = 1 + 0.0133511x,$$

$$\begin{aligned}
y_1 &= -0.0133615x + 0.333333x^3 + 0.00185432x^4 + 0.0416667x^6 + 0.000353204x^7 + 0.00126263x^9 + 0.0000123622x^{10}, \\
y_2 &= 0.000010409x - 0.00185577x^4 + 0.0138889x^6 - 0.000304423x^7 + 0.00473485x^9 + 8.06829 \cdot 10^{-6}x^{10} + \\
&0.000412677x^{12} + 2.3547 \cdot 10^{-6}x^{13} + 0.0000152081x^{15} + 1.10376 \cdot 10^{-7}x^{16} + 2.06311 \cdot 10^{-7}x^{18} + 1.72127 \cdot 10^{-9}x^{19}, \\
y_3 &= -8.88195 \cdot 10^{-9}x + 1.4457 \cdot 10^{-6}x^4 - 0.000048819x^7 + 0.000175365x^9 - 0.0000199921x^{10} + \\
&0.000100778x^{12} - 2.05954 \cdot 10^{-6}x^{13} + 0.0000183093x^{15} - 4.63411 \cdot 10^{-8}x^{16} + 1.47283 \cdot 10^{-6}x^{18} + 4.5557 \cdot 10^{-9}x^{19} + \\
&5.86771 \cdot 10^{-8}x^{21} + 3.13463 \cdot 10^{-10}x^{22} + 1.2034 \cdot 10^{-9}x^{24} + 7.87587 \cdot 10^{-12}x^{25} + 1.01342 \cdot 10^{-11}x^{27} + 7.58937 \cdot 10^{-14}x^{28},
\end{aligned}$$

the series solution by L_2 is

$$\begin{aligned}
y &= y_0 + y_1 + y_2 + y_3 = \\
&1 - 7.63641 \cdot 10^{-12}x + 0.333333x^3 + 1.23254 \cdot 10^{-9}x^4 + 0.0555556x^6 - 3.79787 \cdot 10^{-8}x^7 + 0.00617284x^9 + 4.3831 \cdot 10^{-7}x^{10} + \\
&0.000513454x^{12} + 2.95158 \cdot 10^{-7}x^{13} + 0.0000335174x^{15} + 6.40353 \cdot 10^{-8}x^{16} + 1.67914 \cdot 10^{-6}x^{18} + 6.27697 \cdot 10^{-9}x^{19} + \\
&5.86771 \cdot 10^{-8}x^{21} + 3.13463 \cdot 10^{-10}x^{22} + 1.2034 \cdot 10^{-9}x^{24} + 7.87587 \cdot 10^{-12}x^{25} + 1.01342 \cdot 10^{-11}x^{27} + 7.58937 \cdot 10^{-14}x^{28}.
\end{aligned}$$

The result by L_3

$$y_0 = 1 + 0.0133511x,$$

As we noted y_0 equal y_0 in L_2 , then the solutions by L_2 and L_3 are same.

The result by L_4

$$y_0 = 1,$$

As we noted y_0 equal y_0 in L_1 , then the solutions by L_1 and L_4 are same.

The result by L_5

$$y_0 = 1,$$

As we noted y_0 equal y_0 in L_1, L_4 , then the solutions by L_1, L_4 and L_5 are same.

The result by L_6

$$y_0 = 1 + 0.00555704x,$$

$$\begin{aligned}
y_1 &= -0.00555821x + 0.333333x^3 + 0.000771811x^4 + 0.0416667x^6 + 0.000147012x^7 + 0.00126263x^9 + 5.1454 \cdot 10^{-6}x^{10}, \\
y_2 &= 1.17378 \cdot 10^{-6}x - 0.000771974x^4 + 0.0138889x^6 - 0.000126624x^7 + 0.00473485x^9 + 3.36112 \cdot 10^{-6}x^{10} + \\
&0.000412677x^{12} + 9.80077 \cdot 10^{-7}x^{13} + 0.0000152081x^{15} + 4.59411 \cdot 10^{-8}x^{16} + 2.06311 \cdot 10^{-7}x^{18} + 7.16431 \cdot 10^{-10}x^{19}, \\
y_3 &= -2.71278 \cdot 10^{-10}x + 1.63025 \cdot 10^{-7}x^4 - 0.0000203915x^7 + 0.000175365x^9 - 8.31927 \cdot 10^{-6}x^{10} + \\
&0.000100778x^{12} - 8.56669 \cdot 10^{-7}x^{13} + 0.0000183093x^{15} - 1.92621 \cdot 10^{-8}x^{16} + 1.47283 \cdot 10^{-6}x^{18} + \\
&1.89659 \cdot 10^{-9}x^{19} + 5.86771 \cdot 10^{-8}x^{21} + 1.3047 \cdot 10^{-10}x^{22} + 1.2034 \cdot 10^{-9}x^{24} + 3.27811 \cdot 10^{-12}x^{25} + \\
&1.01342 \cdot 10^{-11}x^{27} + 3.15887 \cdot 10^{-14}x^{28},
\end{aligned}$$

the series solution by L_6 is

$$\begin{aligned}
y &= y_0 + y_1 + y_2 + y_3 = \\
&1 - 6.25559 \cdot 10^{-14}x + 0.333333x^3 + 3.76688 \cdot 10^{-11}x^4 + 0.0555556x^6 - 4.30467 \cdot 10^{-9}x^7 + 0.00617284x^9 + \\
&1.87262 \cdot 10^{-7}x^{10} + 0.000513454x^{12} + 1.23408 \cdot 10^{-7}x^{13} + 0.0000335174x^{15} + 2.6679 \cdot 10^{-8}x^{16} + 1.67914 \cdot 10^{-6}x^{18} + \\
&2.61302 \cdot 10^{-9}x^{19} + 5.86771 \cdot 10^{-8}x^{21} + 1.3047 \cdot 10^{-10}x^{22} + 1.2034 \cdot 10^{-9}x^{24} + 3.27811 \cdot 10^{-12}x^{25} + \\
&1.01342 \cdot 10^{-11}x^{27} + 3.15887 \cdot 10^{-14}x^{28}.
\end{aligned}$$

In our analysis, we establish that y_0 in both operators L_1 , L_4 and L_5 is equivalent. This equivalence signifies that the solutions generated by L_1 , L_4 and L_5 are fundamentally the same. The implication of this finding is significant, as it validates the consistency of our modified operators within the framework of the Modified Adomian Decomposition Method. By ensuring that y_0 remains unchanged across different operators, we reinforce the reliability and accuracy of the solutions obtained. This consistency not only strengthens our methodology but also enhances confidence in the applicability of the MADM in addressing the complexities of the third-order Emden-Fowler equations.

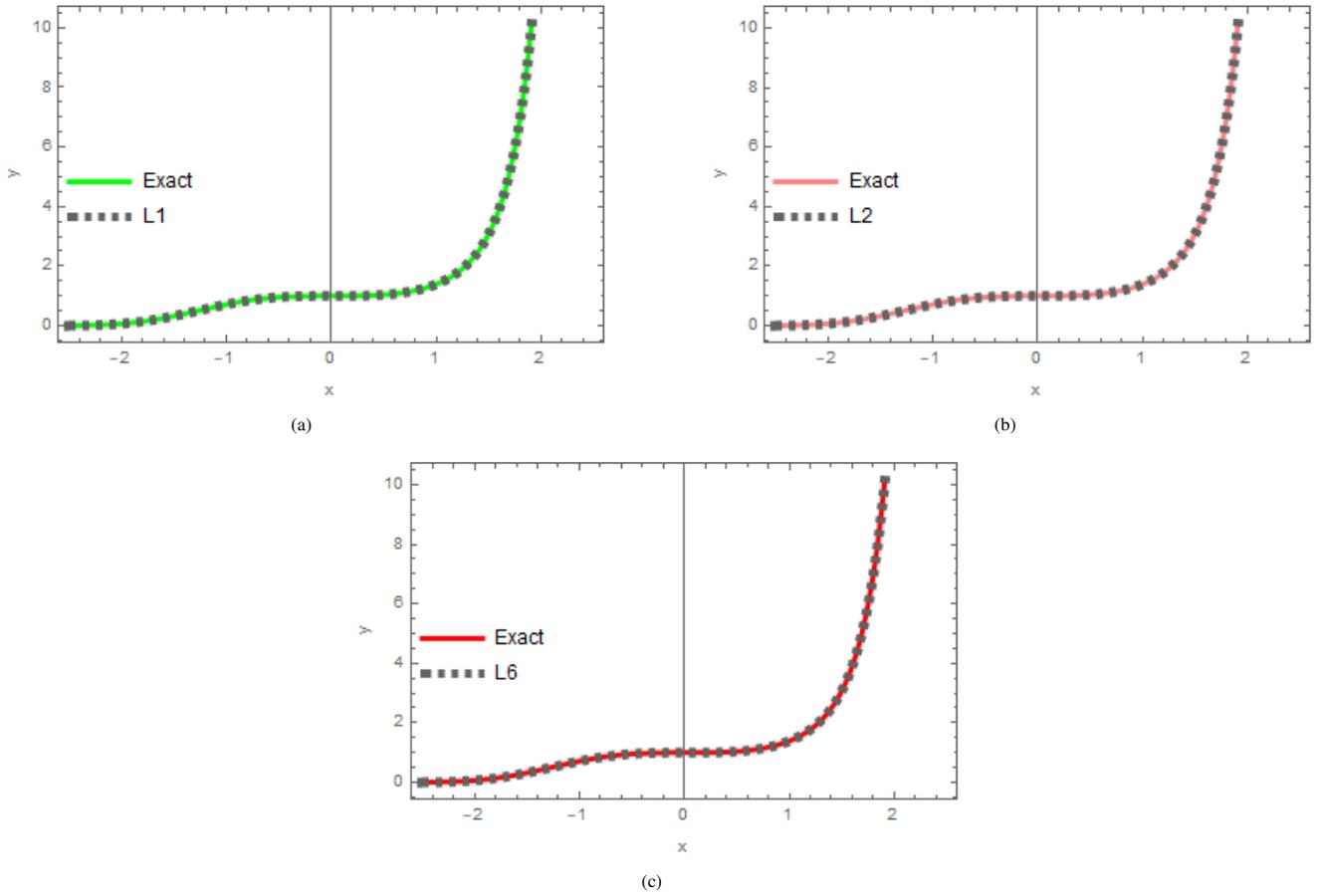


Figure 1: Comparing the between the exact solution and by MADM(L_1, L_2, L_6)

x	Exact solution	MADM L_1	MADM L_2	MADM L_6	MLPNN [20]
0	1	1	1	1	1
0.1	1.000333	1.000333	1.000333	1.000333	1.000334
0.2	1.002670	1.002670	1.002670	1.002670	1.002672
0.3	1.009041	1.009041	1.009041	1.009041	1.009042
0.4	1.021563	1.021563	1.021563	1.021563	1.021562
0.5	1.042547	1.042547	1.042547	1.042547	1.042546
0.6	1.074655	1.074655	1.074655	1.074655	1.074655
0.7	1.121126	1.121126	1.121126	1.121126	1.121126
0.8	1.186095	1.186095	1.186095	1.186095	1.186090
0.9	1.275069	1.275068	1.275068	1.275068	1.275065
1.0	1.395612	1.395610	1.395611	1.395611	1.395620

Table 1: Compare the results obtained by MADM (L_1, L_2, L_6), MLPNN and the exact solution .

x	Absolute Error			
	MADM L1	MADM L2	MADM L6	MLPNN [20]
0	0	0	0	0
0.1	5.0×10^{-17}	6.0×10^{-13}	2.0×10^{-15}	1.0×10^{-6}
0.2	4.0×10^{-15}	4.0×10^{-17}	-8.0×10^{-15}	2.0×10^{-6}
0.3	5.0×10^{-13}	-2.0×10^{-12}	5.0×10^{-14}	1.0×10^{-6}
0.4	2.0×10^{-11}	3.0×10^{-12}	2.0×10^{-12}	1.0×10^{-6}
0.5	3.0×10^{-10}	1.0×10^{-11}	9.0×10^{-11}	1.0×10^{-6}
0.6	2.0×10^{-9}	3.0×10^{-10}	1.0×10^{-9}	1.0×10^{-6}
0.7	2.0×10^{-8}	5.0×10^{-9}	1.0×10^{-8}	2.0×10^{-6}
0.8	9.0×10^{-8}	4.0×10^{-8}	7.0×10^{-8}	5.0×10^{-6}
0.9	5.0×10^{-7}	2.0×10^{-7}	4.0×10^{-7}	3.0×10^{-6}
1.0	2.0×10^{-6}	1.0×10^{-6}	2.0×10^{-6}	7.0×10^{-6}

Table 2: Absolute error between Exact solution, MADM (L_1, L_2, L_6) and MLPNN

The modifications presented in this article for solving third-order Emden-Fowler equations with initial and boundary conditions yield results that are very close to the exact solution. As demonstrated in Table 1, a comparison between the exact solution and the modifications L_1, L_2 and L_6 highlights that our results obtained through L_1, L_2 and L_6 closely approximate the exact solution. Table 2. further illustrates the absolute errors between the exact solution and the modifications L_1, L_2 and L_5 , revealing that the absolute errors for both methods are very small. Additionally, Figure 1 shows that the modifications converge to the exact solution, further underscoring their effectiveness in solving these equations accurately.

Example 2. Consider the non-linear third-order Emden-Fowler equation [20]

$$y''' + \frac{2}{x}y'' = \frac{9}{8}(x^6 + 8)y^{-5}, \quad 0 \leq x \leq 1 \tag{10}$$

with initial and boundary conditions

$$y(0) = 1, y'(0) = 0, y''(0) = 0, \\ y(1) = 1, y(0.1) = 1.06066, y'(0) = 0,$$

Eq.(10) rewards the following equation

$$Ly = \frac{9}{8}(x^6 + 8)y^{-5}, \tag{11}$$

putting $c = 2$ in Eq.(3) we get three operators. Where $L_1 = L_6, L_2 = L_3$ and $L_4 = L_5$.

$$L_1(y) = L_6(y) = x^{-2} \frac{d}{dx} x^2 \frac{d^2}{dx^2} (y),$$

$$L_2(y) = L_3(y) = x^{-2} \frac{d}{dx} x \frac{d}{dx} x^2 \frac{d}{dx} x^{-1} (y),$$

$$L_4(y) = L_5(y) = x^{-1} \frac{d^2}{dx^2} x \frac{d}{dx} (y),$$

the inverse integrations are

$$L_1^{-1} = \int_0^x \int_0^x x^{-2} \int_0^x x^2 (y) dx dx dx,$$

$$L_2^{-1}(y) = x \int_{0.1}^x x^{-2} \int_0^x x^{-1} \int_0^x x^2 (y) dx dx dx,$$

$$L_4^{-1}(y) = \int_0^x x^{-1} \int_0^x \int_0^x x (y) dx dx dx,$$

Taking L^{-1} on both sides of Eq.(11), and using steps in section 2. We get the following results.

The result by using L_1

$$y_0 = 1, \\ y_1 = 0.5x^3 + 0.00173611x^9, \\ y_2 = -0.125x^6 - 0.00182489x^{12} - 1.77299 \cdot 10^{-6}x^{18}, \\ y_3 = 0.0607639x^9 + 0.00166297x^{15} + 4.54065 \cdot 10^{-6}x^{21} + 3.20965 \cdot 10^{-9}x^{27}, \\ y_4 = -0.0372376x^{12} - 0.00148202x^{18} - 7.65899 \cdot 10^{-6}x^{24} - 1.31429 \cdot 10^{-8}x^{30} - 7.23062 \cdot 10^{-12}x^{36},$$

the series solution by L_1 is

$$y = y_0 + y_1 + y_2 + y_3 + y_4 = \\ 1. + 0.5x^3 - 0.125x^6 + 0.0625x^9 - 0.0390625x^{12} + 0.00166297x^{15} - 0.0014838x^{18} + \\ 4.54065 \cdot 10^{-6}x^{21} - 7.65899 \cdot 10^{-6}x^{24} + 3.20965 \cdot 10^{-9}x^{27} - 1.31429 \cdot 10^{-8}x^{30} - 7.23062 \cdot 10^{-12}x^{36},$$

the result by using L_2

$$y_0 = 1 + 0.00499875x,$$

$$\begin{aligned}
 y_1 &= -0.00499532x + 0.5x^3 - 0.00468633x^4 + 0.0000337331x^5 - 2.18586 \cdot 10^{-7}x^6 + 1.33795 \cdot 10^{-9}x^7 - \\
 &\quad 7.90028 \cdot 10^{-12}x^8 + 0.00173611x^9 + \dots, \\
 y_2 &= -3.42938 \cdot 10^{-6}x + 0.00468311x^4 - 0.0000674199x^5 - 0.124999x^6 + 0.00301263x^7 - 0.0000438606x^8 + \\
 &\quad 4.99083 \cdot 10^{-7}x^9 + \dots, \\
 y_3 &= -4.17604 \cdot 10^{-9}x + 3.21505 \cdot 10^{-6}x^4 + 0.0000336405x^5 - 6.54408 \cdot 10^{-7}x^6 - 0.00301056x^7 + \\
 &\quad 0.0000876609x^8 + 0.0607624x^9 + \dots, \\
 y_4 &= -6.25623 \cdot 10^{-12}x - 3.87741 \cdot 10^{-26}x^2 - 4.65289 \cdot 10^{-25}x^3 + 3.91504 \cdot 10^{-9}x^4 + 4.62533 \cdot 10^{-8}x^5 + \\
 &\quad 2.18136 \cdot 10^{-7}x^6 - 2.06682 \cdot 10^{-6}x^7 - 0.000043786x^8 + 3.56259 \cdot 10^{-7}x^9 + \dots,
 \end{aligned}$$

the series solution by L_2 is

$$\begin{aligned}
 y &= y_0 + y_1 + y_2 + y_3 + y_4 = \\
 &1 + 1.27138 \cdot 10^{-14}x - 3.87741 \cdot 10^{-26}x^2 + 0.5x^3 - 1.80293 \cdot 10^{-11}x^4 - 7.51132 \cdot 10^{-12}x^5 - 0.125x^6 + 6.24642 \cdot 10^{-9}x^7 + \\
 &\quad 1.42742 \cdot 10^{-8}x^8 + 0.0624994x^9 + \dots
 \end{aligned}$$

The result by using L_4

$$\begin{aligned}
 y_0 &= 1, \\
 y_1 &= 0.5x^3 + 0.00173611x^9, \\
 y_2 &= -0.125x^6 - 0.00182489x^{12} - 1.77299 \cdot 10^{-6}x^{18}, \\
 y_3 &= 0.0607639x^9 + 0.00166297x^{15} + 4.54065 \cdot 10^{-6}x^{21} + 3.20965 \cdot 10^{-9}x^{27}, \\
 y_4 &= -0.0372376x^{12} - 0.00148202x^{18} - 7.65899 \cdot 10^{-6}x^{24} - 1.31429 \cdot 10^{-8}x^{30} - 7.23062 \cdot 10^{-12}x^{36},
 \end{aligned}$$

the series solution by L_4 is

$$\begin{aligned}
 y &= y_0 + y_1 + y_2 + y_3 + y_4 = \\
 &1. + 0.5x^3 - 0.125x^6 + 0.0625x^9 - 0.0390625x^{12} + 0.00166297x^{15} - 0.0014838x^{18} + \\
 &\quad 4.54065 \cdot 10^{-6}x^{21} - 7.65899 \cdot 10^{-6}x^{24} + 3.20965 \cdot 10^{-9}x^{27} - 1.31429 \cdot 10^{-8}x^{30} - 7.23062 \cdot 10^{-12}x^{36},
 \end{aligned}$$

As we noted the results by L_1 and L_4 are same.

x	Exact solution	MADM $L_1 = L_4$	MADM L_2	MLPNN [20]
0	1	1	1	1
0.1	1.000500	1.000500	1.000500	1.000486
0.2	1.003992	1.003992	1.003992	1003957
0.3	1.013410	1.013410	1.013410	1.013478
0.4	1.031504	1.031504	1.031504	1.032031
0.5	1.060660	1.060659	1.060659	1.062344
0.6	1.102724	1.102713	1.102714	1.106744
0.7	1.158879	1.158707	1.158784	1.167028
0.8	1.22963	1.228968	1.2289988	1.244348
0.9	1.314914	1.311371	1.311491	1.339113
1.0	1.414214	1.398614	1.399373	1.450907

Table 3: Compare the results obtained by MADM ($L_1 = L_4, L_2$), MLPNN and the exact solution.

x	Absolute Error		
	MADM L_1	MADM L_2	MLPNN [20]
0	0	0	0
0.1	0	2.0×10^{-16}	0.1×10^{-5}
0.2	8.0×10^{-13}	9.0×10^{-14}	3.5×10^{-5}
0.3	4.0×10^{-10}	2.0×10^{-10}	6.8×10^{-5}
0.4	3.0×10^{-8}	2.0×10^{-8}	5.3×10^{-4}
0.5	7.0×10^{-7}	6.0×10^{-7}	1.7×10^{-3}
0.6	1.0×10^{-5}	10.0×10^{-6}	4.02×10^{-3}
0.7	10.0×10^{-5}	9.0×10^{-5}	8.1×10^{-3}
0.8	7.0×10^{-4}	7.0×10^{-4}	1.5×10^{-2}
0.9	3.0×10^{-3}	3.0×10^{-3}	2.4×10^{-2}
1.0	2.0×10^{-2}	1.0×10^{-2}	3.7×10^{-2}

Table 4: Absolute error between Exact solution, MADM ($L_1 = L_4, L_2$) and MLPNN

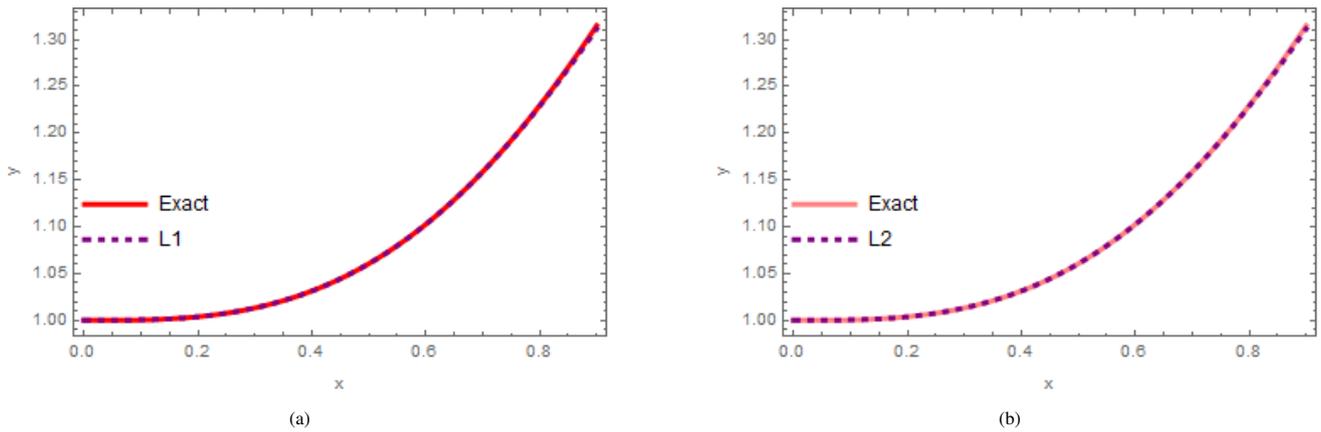


Figure 2: Compare between Exact solution and MADM (L_1, L_2)

To validate the accuracy of the proposed MADM, the results for Example 1, and Example 2 are compared with the method reported in [20]. As shown in the results, our method achieves the exact solution with an absolute error of approximately 10^{-15} . In contrast, the method in [20] provides a numerical approximation that requires extensive training and optimization. This proves that the MADM is more efficient and provides superior precision for singular third-order problems

Example 3. Consider the following equation [20]

$$y''' - \frac{2}{x}y'' = y^3(x) - 6e^x + 6e^x x + 7e^x x^2 + e^x x^3 + e^{3x} x^9, \tag{12}$$

with boundary conditions

$$y(0) = 0, y'(0) = 0, y''(0.1) = 0.730518,$$

$$y(0) = 0, y(0.2) = 1.00267, y''(0.1) = 0.730518,$$

$$y(0.1) = 0.00110517, y(0.2) = 0.00977122, y'(0.2) = 0.15634,$$

$$y(0) = 0, y'(0) = 0, y'(0.1) = 0.342603,$$

$$y(0) = 0, y'(0) = 0, y(0.1) = 0.00110517,$$

Eq.(12) can be write as

$$Ly = y^3(x) - 6e^x + 6e^x x + 7e^x x^2 + e^x x^3 + e^{3x} x^9, \tag{13}$$

putting $c = -2$ in Eq.(3) and Eq.(4) we get the following differential operators and it's inverse operators.

$$L_1(y) = x^2 \frac{d}{dx} x^2 - \frac{d^2}{dx^2} (y),$$

$$L_2(y) = x^2 \frac{d}{dx} x^{-3} \frac{d}{dx} x^2 \frac{d}{dx} x^{-1} (y),$$

$$L_3(y) = x^{-2} \frac{d}{dx} x^5 \frac{d}{dx} x^{-2} \frac{d}{dx} x^{-1} (y),$$

$$L_4(y) = x^{-1} \frac{d}{dx} x^4 \frac{d}{dx} x^{-3} \frac{d}{dx} (y),$$

$$L_5(y) = x^{-1} \frac{d^2}{dx^2} x^5 \frac{d}{dx} x^{-4} (y),$$

$$L_6(y) = x^{-2} \frac{d}{dx} x^2 \frac{d}{dx} x^4 \frac{d}{dx} x^{-4} (y),$$

$$L_1^{-1} = \int_0^x \int_0^x x^2 \int_0.1^x x^{-2} (y) dx dx dx,$$

$$L_2^{-1} (y) = x \int_0.2^x x^{-2} \int_0^x x^3 \int_0.1^x x^{-2} (y) dx dx dx,$$

$$L_3^{-1} (y) = x \int_0.1^x x^2 \int_0.2^x x^{-5} \int_0^x x^2 (y) dx dx dx,$$

$$L_4^{-1} (y) = \int_0^x x^3 \int_0.1^x x^{-4} \int_0^x x (y) dx dx dx,$$

$$L_5^{-1} (y) = x^4 \int_0.1^x x^{-5} \int_0^x \int_0^x x (y) dx dx dx,$$

$$L_6^{-1} (y) = x^4 \int_0.1^x x^{-4} \int_0.2^x x^{-2} \int_0^x x^2 (y) dx dx dx.$$

Taking L^{-1} on both sides of Eq.(13), and Applying the steps in section (2), we get the following results.

The result by L_1 is

$$y_0 = 1.x^3 + 1.x^4 + 0.5.x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + 0.00138889x^9 + 0.000198413x^{10} + 0.0000248016x^{11} + \dots,$$

$$y_1 = -1.3606 \cdot 10^{-10} x^4 + 0.00094697 x^{12} + 0.00213675 x^{13} + 0.00247253 x^{14} + 0.00194805 x^{15} + \dots,$$

$$y_2 = 4.31722 \cdot 10^{-21} x^4 - 2.90727 \cdot 10^{-13} x^{13} - 4.4855 \cdot 10^{-13} x^{14} - 3.53403 \cdot 10^{-13} x^{15} + \dots,$$

the solution by L_1 is

$$y = y_0 + y_1 + y_2 =$$

$$1 \cdot x^3 + 1 \cdot x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + 0.00138889x^9 + 0.000198413x^{10} + 0.0000248016x^{11} +$$

$$0.00094697x^{12} + 0.00213675x^{13} + 0.00247253x^{14} + 0.00194805x^{15} + \dots$$

The result by L_2 is

$$y_0 = 5.5128 \cdot 10^{-14} x + 1 \cdot x^3 + 1 \cdot x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + 0.00138889x^9 +$$

$$0.000198413x^{10} + 0.0000248016x^{11},$$

$$y_1 = -2.94447 \cdot 10^{-11} x - 1.3606 \cdot 10^{-10} x^4 + 2.79232 \cdot 10^{-42} x^6 + 4.07022 \cdot 10^{-29} x^8 + 2.53258 \cdot 10^{-29} x^9 +$$

$$3.06267 \cdot 10^{-16} x^{10} + 4.29569 \cdot 10^{-16} x^{11} + \dots,$$

$$y_2 = 9.22822 \cdot 10^{-20} x + 1.46105 \cdot 10^{-18} x^4 - 4.47427 \cdot 10^{-39} x^6 - 4.34794 \cdot 10^{-26} x^8 - 2.70538 \cdot 10^{-26} x^9 -$$

$$1.63582 \cdot 10^{-13} x^{10} - 2.2944 \cdot 10^{-13} x^{11} + \dots,$$

the solution by L_2 is

$$y = y_0 + y_1 + y_2 =$$

$$-2.93896 \cdot 10^{-11} x + 1 \cdot x^3 + 1 \cdot x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + 0.00138889x^9 +$$

$$0.000198413x^{10} + 0.0000248016x^{11} + \dots$$

The result by L_3 is

$$y_0 = -3.14881 \cdot 10^{-8} x + 1 \cdot x^3 + 1.00003x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + \dots,$$

$$y_1 = 1.45633 \cdot 10^{-11} x - 1.45751 \cdot 10^{-8} x^4 - 5.20339 \cdot 10^{-25} x^6 + 1.3279 \cdot 10^{-17} x^8 + 8.26275 \cdot 10^{-18} x^9 -$$

$$1.74934 \cdot 10^{-10} x^{10} - 2.45369 \cdot 10^{-10} x^{11} + \dots,$$

$$y_2 = -6.04693 \cdot 10^{-20} x + 6.04125 \cdot 10^{-17} x^4 + 7.21974 \cdot 10^{-28} x^6 - 1.22831 \cdot 10^{-20} x^8 - 7.6432 \cdot 10^{-21} x^9 +$$

$$8.09072 \cdot 10^{-14} x^{10} + 1.13488 \cdot 10^{-13} x^{11} + \dots,$$

the solution by L_3 is

$$y = y_0 + y_1 + y_2 =$$

$$3.14735 \cdot 10^{-8} x + 1 \cdot x^3 + 1.00003x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 1.32667 \cdot 10^{-17} x^8 + 8.2551 \cdot 10^{-18} x^9 -$$

$$1.74853 \cdot 10^{-10} x^{10} - 2.45256 \cdot 10^{-10} x^{11} + \dots$$

The result by L_4 is

$$y_0 = 1 \cdot x^3 + 1 \cdot x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + \dots,$$

$$y_1 = -3.62969 \cdot 10^{-11} x^4 + 0.00094697 x^{12} + 0.00213675 x^{13} + 0.00247253 x^{14} + 0.00194805 x^{15} + \dots,$$

$$y_2 = 2.67028 \cdot 10^{-22} x^4 - 7.75575 \cdot 10^{-14} x^{13} - 1.1966 \cdot 10^{-13} x^{14} - 9.42777 \cdot 10^{-14} x^{15} + \dots,$$

the solution by L_4 is

$$y = y_0 + y_1 + y_2 =$$

$$1 \cdot x^3 + 1 \cdot x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + 0.00094697x^{12} + 0.00213675x^{13} +$$

$$0.00247253x^{14} + 0.00194805x^{15} + \dots$$

The result by L_5 is

$$y_0 = 1 \cdot x^3 + 0.999991x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + \dots,$$

$$y_1 = -1.18744 \cdot 10^{-11} x^4 + 0.00094697 x^{12} + 0.00213673 x^{13} + 0.0024725 x^{14} + 0.00194803 x^{15} + \dots,$$

$$y_2 = 2.38731 \cdot 10^{-23} x^4 - 2.53726 \cdot 10^{-14} x^{13} - 3.9146 \cdot 10^{-14} x^{14} - 3.08423 \cdot 10^{-14} x^{15} + \dots,$$

the solution by L_5 is

$$y = y_0 + y_1 + y_2 =$$

$$1 \cdot x^3 + 0.999991x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8 + 0.00094697x^{12} + 0.00213673x^{13} +$$

$$0.0024725x^{14} + 0.00194803x^{15} + \dots$$

The result by L_6 is

$$y_0 = -1.48357 \cdot 10^{-7} x + 1 \cdot x^3 + 1.00015x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + \dots,$$

$$y_1 = 8.6071 \cdot 10^{-11} x - 8.60829 \cdot 10^{-8} x^4 - 5.44222 \cdot 10^{-23} x^6 + 2.94776 \cdot 10^{-16} x^8 + 1.83443 \cdot 10^{-16} x^9 + \dots,$$

$$y_2 = -2.45539 \cdot 10^{-18} x + 2.45505 \cdot 10^{-15} x^4 + 9.47208 \cdot 10^{-26} x^6 - 3.42034 \cdot 10^{-19} x^8 - 2.12869 \cdot 10^{-19} x^9 + \dots,$$

the solution by L_6 is

$$y = y_0 + y_1 + y_2 =$$

$$-1.48271 \cdot 10^{-7} x + 1 \cdot x^3 + 1.00015x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 2.94434 \cdot 10^{-16} x^8 + 1.8323 \cdot 10^{-16} x^9 + \dots$$

In Examples 2 and 3, we utilize Figures 2 and 3 to illustrate the convergence behavior of the Modified Adomian Decomposition Method (MADM). These figures visually represent how the solutions generated by MADM approach the exact solutions of the equations, thereby affirming the method's reliability and effectiveness. Additionally, in Example 2, we employ Table 3 to facilitate a comparison between the exact solutions and those obtained through MADM. This table provides a clear numerical representation that allows for an assessment of the accuracy and performance of our method. Furthermore, Table 4 is introduced to detail the absolute errors between the exact solutions and the MADM-derived solutions in Example 2. By analyzing these absolute errors, we gain insights into the precision of our approach and can effectively quantify the discrepancies between the solutions. This comprehensive analysis highlights the robustness of the MADM in yielding accurate results for the third-order Emden-Fowler equations.

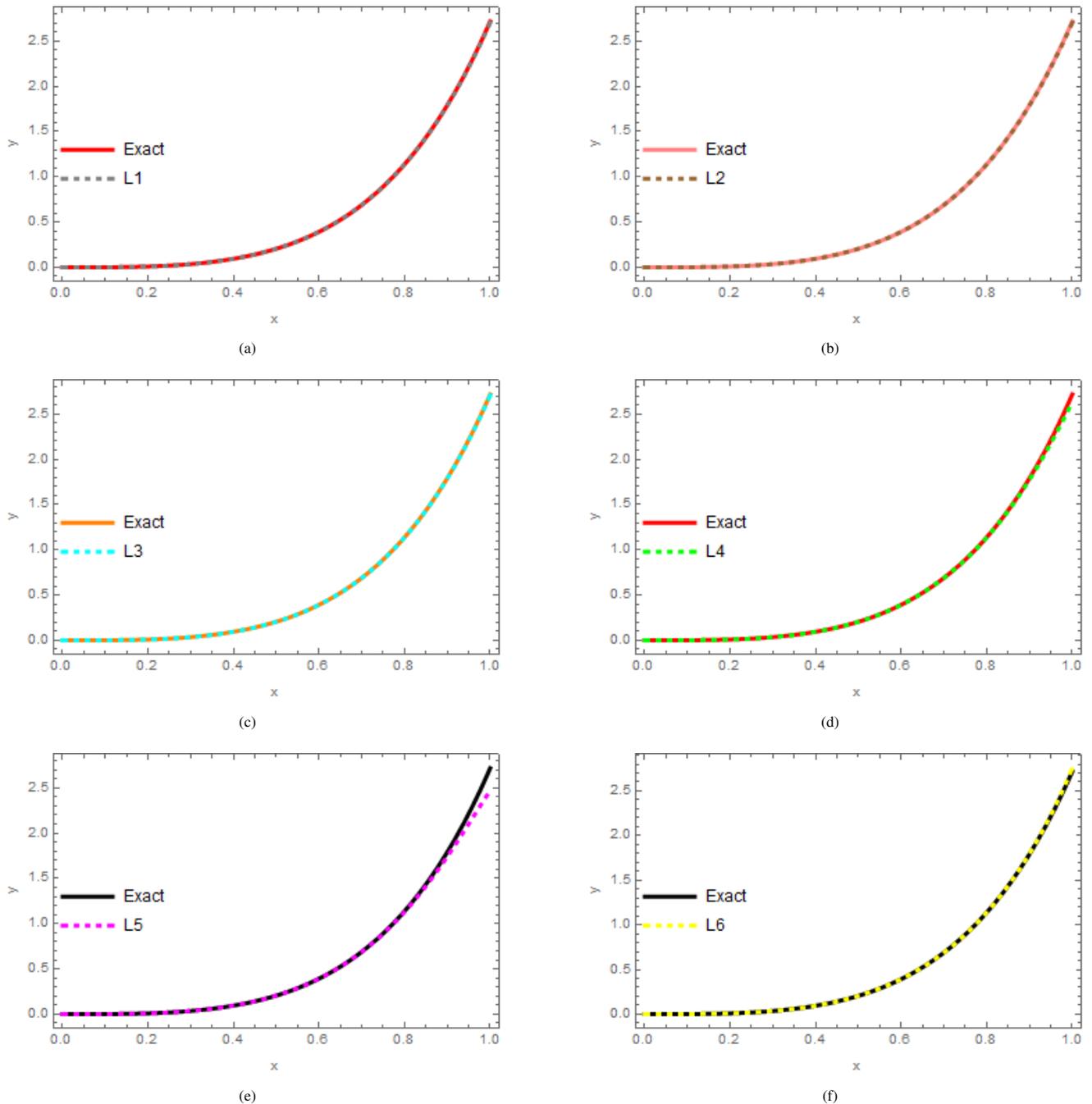


Figure 3: Compare between Exact solution and MADM ($L_1, L_2, L_3, L_4, L_5, L_6$)

5. conclusion

In this article, we introduced a new modified method for the Adomian decomposition, illustrating its effectiveness through the analysis of both linear and nonlinear examples under various initial and boundary conditions. In our approach, we specifically examined three examples, allowing us to demonstrate the method's capabilities in diverse scenarios. In the first linear example, we set the constant $c = 6$ in equation 1, which resulted in the derivation of six differential operators. This set of operators enabled us to solve the corresponding equation, yielding results that were significantly close to the exact solution. This success validated the potential of the modified method in handling linear differential equations. Following the linear case, we analyzed two nonlinear examples where the values of c were set to $c = 2$ and $c = -2$ respectively. In these instances, we again derived six differential operators. Notably, in the second nonlinear example, we found that certain operators were equal among themselves: specifically, we observed that L_1 and L_6 were equal, as were L_2 and L_3 , as well as L_4 and L_5 . This property not only reinforces the consistency of our approach but also highlights the structure provided by the modifications. Throughout all examples analyzed, we demonstrated the convergence between the exact solutions and our newly proposed modifications. The results obtained were very close to the exact solutions, indicating the high efficiency of the modifications introduced in this study. This efficiency suggests that our method not only provides accurate solutions but also enhances the reliability of the Adomian decomposition technique. The modifications presented in this article significantly improve the Adomian decomposition method's capability to solve both linear and nonlinear differential equations effectively. The analysis of the examples confirmed that the method produces results closely aligning with

exact solutions, proving its efficiency and robustness. Thus, we recommend the application of the modified Adomian decomposition method for solving a broader range of differential equations, given its demonstrated accuracy and effectiveness in this study.

While the proposed MADM shows high efficiency and accuracy, its performance may vary with highly oscillating nonlinear terms. Future work will focus on extending these modified operators to handle partial differential equations (PDEs) and systems of nonlinear singular equations. Further research into the stability analysis of these operators is also recommended as an open problem.

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