Global Journal of Mathematical Analysis, 12 (1) (2026) 1-6

Global Journal of Mathematical Analysis

Website: www.sciencepubco.com/index.php/GJMA
doi: https://doi.org/10.14419/hgz0m151
Research paper

Fixed Point Results for θ-Rational Contraction in Complete Metric Spaces

Anas Yusuf[1*] **and Almustapha Umar**[2]

[1,2]*Department of Mathematics, Federal University, Birnin Kebbi, Kebbi State, Nigeria*
[*]*Corresponding author E-mail:anas.yusuf@fubk.edu.ng*

Abstract

This paper introduces and investigates a new family of nonlinear contraction-type mappings, termed θ-rational contractions, defined on complete metric spaces. This class is formulated through a function $\theta \in \Theta$ satisfying appropriate monotonicity and asymptotic regularity conditions, and incorporates both rational-type expressions and power-type nonlinearities. The proposed framework unifies and extends several well-known contractive conditions, including the Jleli-Samet θ-contractions and the rational-type contractions of Dass-Gupta. A general fixed point theorem is established, ensuring that a fixed point exist and is unique for θ-rational contractions, together with the convergence of the associated Picard iteration. A corollary shows that, by choosing $a = 0$, the contractive condition reduces to a strict θ-contraction, thereby generalizing and strengthening the θ-contraction introduced by Jleli and Samet. Illustrative examples on both finite and continuous metric spaces are included to show the applicability of the established findings. Additionally, a common fixed point result is established for commuting pairs of θ-rational contractions. These results highlight the flexibility of the θ-rational framework and offer a unified approach to a broad class of nonlinear contractive conditions.

Keywords: *Complete metric space; Fixed point theorem; Nonlinear contractive mapping; Picard iteration; θ-rational contraction.*

1. Introduction

The investigation of fixed points for nonlinear mappings in metric spaces has long been central to analysis and its applications. Since the appearance of Banach's classical contraction principle in his 1922 doctoral thesis [1], where it was employed to show that the solutions to integral equations exist, a vast body of generalizations has emerged. These extensions were developed to treat mappings that fail to satisfy the simple Lipschitz-type condition required in Banach's theorem. Among the most prominent are the contractions of Kannan [2], Chatterjea [3], Hardy–Rogers [4], and, more recently, the θ-contractions formulated by Jleli and Samet [5]. Their approach relies on a nonlinear transformation of the metric via a distortion function θ belonging to the class Θ, where Θ denotes the set of functions $\theta : (0,\infty) \to (1,\infty)$ that satisfy the following properties:

- (Θ_1) θ is non-decreasing;
- (Θ_2) for every sequence $\{t_n\}$, with $\lim_{n\to\infty} t_n = 0^+$, we have

  $$\lim_{n\to\infty} \theta(t_n) = 1;$$

- (Θ_3) There exist $k \in (0,1)$, $l \in (0,\infty]$ such that

  $$\lim_{t\to 0^+} \frac{\theta(t)-1}{t^k} = l.$$

A mapping $\mathfrak{T} : U \to U$ on a generalized metric space (U,ρ) is called a θ-contraction if there exist $u,v \in U$ with $\rho(\mathfrak{T}u,\mathfrak{T}v) \neq 0$, and a constant $r \in (0,1)$ such that for all $u,v \in U$,

$$\theta(\rho(\mathfrak{T}u,\mathfrak{T}v)) \le [\theta(\rho(u,v))]^r.$$

Subsequent extensions were established by Jleli, Karapınar, and Samet [6], and numerous fixed point results for θ-contractions have appeared in both generalized and classical metric settings (see [7, 8, 9, 10]).

In parallel with the development of θ-based nonlinear contractions, rational-type contractive conditions and power-type inequalities have become powerful tools for addressing nonlinear problems in which contractive behavior is expressed through rational expressions or powers of the metric. Such conditions arise naturally in nonlinear differential equations, integral equations, and iterative schemes with mixed monotonicity.
Dass and Gupta [11] introduced the rational contractive type conditions to extend Banach's result [1]. They employed the following contractive framework

$$\rho(\mathfrak{T}u,\mathfrak{T}v)\leq a\frac{\rho(v,\mathfrak{T}v)(1+\rho(u,\mathfrak{T}u))}{1+\rho(u,v)}+b\rho(u,v),\quad \forall u,v\in U$$

where $a,b\in[0,1)$ such that $a+b<1$ and $\mathfrak{T}$ continuous, to prove their result. Jaggi [12] later proposed another rational-type condition, namely

$$\rho(\mathfrak{T}u,\mathfrak{T}v)\leq a\frac{\rho(u,\mathfrak{T}u)\rho(v,\mathfrak{T}v)}{\rho(u,v)}+b\rho(u,v),\quad a+b<1,$$

which further generalized Banach's principle. Since then, numerous contributions have enriched the rational contraction theory (see [13, 14, 15, 16, 17, 18]).
The aim of this paper is to unify the ideas of Jleli-Samet [5] and Dass-Gupta [11] by introducing a general θ-rational contraction, which will incorporate simultaneously:

- a rational correction term,
- a powered nonlinear term,
- and the θ-transform, allowing nonlinear geometric distortion.

In contrast to existing frameworks, Classical Banach contractions correspond to linear Lipschitz control, θ-contractions rely solely on nonlinear distortion via θ, while rational contractions involve metric ratios without nonlinear transformation. The present framework unifies these approaches within a single inequality, thereby covering cases not accessible by any of these theories individually.
Our first main theorem establishes the existence and uniqueness of fixed points for such mappings on complete metric spaces, together with Picard convergence of the iterative sequence. As a corollary, we show that taking $a=0$ extend the classical Jleli-Samet θ-contraction result. We also provide illustrative examples on finite and continuous metric spaces, demonstrating the applicability of the theory. Lastly, we obtain a common fixed point theorem for commuting pairs of θ-rational contractions, underscoring the robustness of the framework.
Overall, this work offers a flexible and unifying contraction principle that encompasses many known results as special cases and provides new tools for treating nonlinear problems in fixed point theory.

2. Main Results

Theorem 2.1 (θ-Rational Contraction). *Let (U,ρ) be a complete metric space and let $\theta\in\Theta$. Let constants $a\geq 0$, $b\geq 0$ satisfy $a+b<1$ and fix an exponent $r\in(0,1)$.*
A mapping $\mathfrak{T}:U\to U$ is called a θ-rational contraction if for all $u,v\in U$ with $u\neq v$,

$$\theta\big(\rho(\mathfrak{T}u,\mathfrak{T}v)\big)\leq\frac{a\,\theta(\rho(v,\mathfrak{T}v))\big[1+\theta(\rho(u,\mathfrak{T}u))\big]}{1+\theta(\rho(u,v))}+b\big(\theta(\rho(u,v))\big)^{r}. \tag{1}$$

Then $\mathfrak{T}$ admits a unique fixed point $u^*\in U$, and for each $u_0\in U$ the sequence $\{u_n\}$ defined by $u_{n+1}=\mathfrak{T}u_n$ converges to u^*.

Proof. Fix $u_0\in U$ and define $u_{n+1}=\mathfrak{T}u_n$ for $n\geq 0$. If for some $p\in\mathbb{N}$ we have $u_p=u_{p+1}$, then u_p is a fixed point.Otherwise, assume $\rho(u_n,u_{n+1})>0$ for all n.
Apply (1) with $u=u_{n-1}$ and $v=u_n$. Then $\mathfrak{T}u_{n-1}=u_n$ and $\mathfrak{T}u_n=u_{n+1}$. Hence

$$\begin{aligned}\theta\big(\rho(u_n,u_{n+1})\big)&\leq\frac{a\,\theta(\rho(u_n,u_{n+1}))\big[1+\theta(\rho(u_{n-1},u_n))\big]}{1+\theta(\rho(u_{n-1},u_n))}+b\big(\theta(\rho(u_{n-1},u_n))\big)^{r}\\&=a\,\theta(\rho(u_n,u_{n+1}))+b\big(\theta(\rho(u_{n-1},u_n))\big)^{r}.\end{aligned}\tag{2}$$

Rearranging,

$$(1-a)\,\theta(\rho(u_n,u_{n+1}))\leq b\big(\theta(\rho(u_{n-1},u_n))\big)^{r}.$$

Since $a<1$, the coefficient $1-a$ is strictly positive, allowing us to divide both sides by $1-a$ without altering the inequality. Thus,

$$\theta(\rho(u_n,u_{n+1}))\leq\frac{b}{1-a}\left(\theta(\rho(u_{n-1},u_n))\right)^{r}. \tag{3}$$

Now iterating the above equation (3), we have:

$$\begin{aligned}\theta(\rho(u_n,u_{n+1}))&\leq\left(\frac{b}{1-a}\right)\left(\theta(\rho(u_{n-1},u_n))\right)^{r}.\\&\leq\frac{b}{1-a}\left(\frac{b}{1-a}\left(\theta(\rho(u_{n-2},u_{n-1}))\right)^{r}\right)^{r}\\&\leq\cdots\leq\left(\frac{b}{1-a}\right)^{1+r+\cdots+r^{n-1}}\left(\theta(\rho(u_0,u_1))\right)^{r^n}.\end{aligned}\tag{4}$$

The inequality (4) follows by a straightforward induction on n, by repeatedly applying inequality (3) and then using the geometric sum $1+r+\cdots+r^{n-1} = \frac{1-r^n}{1-r}$.

Since $a+b<1$, we have $\frac{b}{1-a} \in (0,1)$. Hence $\left(\frac{b}{1-a}\right)^{\frac{1-r^n}{1-r}} \longrightarrow \left(\frac{b}{1-a}\right)^{1/(1-r)} < 1,$ as $n \to \infty$, and since $r^n \to 0$, $\big(\theta(\rho(u_0,u_1))\big)^{r^n} \to 1$. Thus, for large n equation (4) is bounded by a number strictly less than 1, so

$$\limsup_{n\to\infty} \theta(\rho(u_n,u_{n+1})) \leq \left(\frac{b}{1-a}\right)^{1/(1-r)} < 1.$$

But $\theta(\rho(u_n,u_{n+1})) \geq 1$ for all n, hence the only possible limit is 1. Thus: $\theta(\rho(u_n,u_{n+1})) \to 1,$ $n \to \infty$. By condition (Θ_2), we have $\lim_{n\to\infty} \rho(u_n,u_{n+1}) = 0$.

From (Θ_3), we have $k \in (0,1)$ such that

$$\lim_{n\to\infty} \frac{\theta(\rho(u_n,u_{n+1}))-1}{(\rho(u_n,u_{n+1}))^k} = l \text{ where } l \in (0,\infty].$$

Case 1: $l<\infty$. Let $F = l/2 > 0$. Using the definition of limit, there exists n_0 such that for all $n \geq n_0$,

$$\left|\frac{\theta(\rho(u_n,u_{n+1}))-1}{(\rho(u_n,u_{n+1}))^k} - l\right| \leq F.$$

This implies, for each $n_0 \leq n$,

$$\frac{\theta(\rho(u_n,u_{n+1}))-1}{(\rho(u_n,u_{n+1}))^k} \geq l - F = F.$$

Then, for each $n \geq n_0$,

$$n(\rho(u_n,u_{n+1}))^k \leq Dn[\theta(\rho(u_n,u_{n+1}))-1], \quad D = 1/F.$$

Case 2: $l = \infty$. Similarly, for any $F>0$, there exists n_0 such that for $n_0 \leq n$

$$\frac{\theta(\rho(u_n,u_{n+1}))-1}{(\rho(u_n,u_{n+1}))^k} \geq l - F = F.$$

Then, for each $n_0 \leq n$,

$$n(\rho(u_n,u_{n+1}))^k \leq Dn[\theta(\rho(u_n,u_{n+1}))-1], \quad D = 1/F.$$

In both cases, there exists $D>0$ and $n_0 \in \mathbb{N}$ such that,

$$n(\rho(u_n,u_{n+1}))^k \leq Dn[\theta(\rho(u_n,u_{n+1}))-1].$$

Using equation (4) we obtain

$$n(\rho(u_n,u_{n+1}))^k \leq Dn\left[\left(\frac{b}{1-a}\right)^{\frac{1-r^n}{1-r}} \big(\theta(\rho(u_0,u_1))\big)^{r^n} - 1\right].$$

Letting $n \to \infty$ gives

$$\lim_{n\to\infty} (\rho(u_n,u_{n+1}))^k = 0.$$

Thus, for some $N \in \mathbb{N}$,

$$\rho(u_n,u_{n+1}) \leq \frac{1}{n^{1/k}}, \qquad n \geq N.$$

Since $k \in (0,1)$, the series $\sum_{n=1}^{\infty} n^{-1/k}$ converges. Therefore by comparison $\sum_{n=1}^{\infty} \rho(u_n,u_{n+1}) < \infty$.

Now, for any $m > n > N$, using the triangular inequality, we have

$$\rho(u_m,u_n) \leq \sum_{i=n}^{m-1} \rho(u_i,u_{i+1}),$$

and the tail of a convergent series goes to zero. Thus:

$$\rho(u_m,u_n) \to 0 \quad \text{as } n,m \to \infty,$$

so (u_n) is Cauchy.

Since U is complete, there exists $u^* \in U$ such that $u_n \to u^*$. We show that u^* is a fixed point.

Apply (1) with $u = u_n$ and $v = u^*$:

$$\theta(\rho(\mathfrak{T}u_n,\mathfrak{T}u^*)) \leq \frac{a\,\theta(\rho(u^*,\mathfrak{T}u^*))[1+\theta(\rho(u_n,\mathfrak{T}u_n))]}{1+\theta(\rho(u_n,u^*))} + b\big(\theta(\rho(u_n,u^*))\big)^r.$$

Let $n \to \infty$, since $u_n \to u^*$ and $\rho(u_n,u^*) \to 0$, we have $\theta(\rho(u_n,u^*)) \to \theta(0^+) = 1$ and $\theta(\rho(u_n,\mathfrak{T}u_n)) = \theta(\rho(u_n,u_{n+1})) \to 1$. Thus the RHS tends to $a\theta(\rho(u^*,\mathfrak{T}u^*)) + b$. Rearranging and passing to the limit gives

$$(1-a)\theta(\rho(\mathfrak{T}u^*,u^*)) \leq b.$$

If $\rho(\mathfrak{T}u^*,u^*) > 0$, then we have $\theta(\rho(\mathfrak{T}u^*,u^*)) > 1$, however

$$\theta(\rho(\mathfrak{T}u^*,u^*)) \leq \frac{b}{1-a} < 1,$$

A contradiction since $\theta \geq 1$. Thus: $\rho(\mathfrak{T}u^*,u^*) = 0$, that is $\mathfrak{T}u^* = u^*$.
To show the uniqueness, suppose v^* is another fixed point of T. Substitute $u = u^*$, $y = v^*$ into (1). Using $\theta(0) = 1$ from (Θ_2), we obtain:

$$\theta(\rho(u^*,v^*)) \leq \frac{a\,\theta(\rho(v^*,\mathfrak{T}v^*))[1+\theta(\rho(u^*,\mathfrak{T}u^*))]}{1+\theta(\rho(u^*,v^*))} + b\big(\theta(\rho(u^*,v^*))\big)^r$$

Thus,

$$\theta(\rho(u^*,v^*)) \leq \frac{a\,\theta(\rho(v^*,v^*))[1+\theta(\rho(u^*,u^*))]}{1+\theta(\rho(u^*,v^*))} + b\big(\theta(\rho(u^*,v^*))\big)^r$$
$$= \frac{a\theta(0)[1+\theta(0)]}{1+\theta(\rho(u^*,v^*))} + b\,\theta(\rho(u^*,v^*))^r.$$

Hence,

$$\theta(\rho(u^*,v^*)) \leq \frac{2a}{1+\theta(\rho(u^*,v^*))} + b\,\theta(\rho(u^*,v^*))^r.$$

Since $r \in (0,1)$, $\theta(\rho(u^*,v^*))^r < \theta(\rho(u^*,v^*))$. Therefore $\theta(\rho(u^*,v^*)) \leq \frac{2a}{1+\theta(\rho(u^*,v^*))} + b\,\theta(\rho(u^*,v^*))$. Rearranging yields:

$$(1-b)\theta(\rho(u^*,v^*)) \leq \frac{2a}{1+\theta(\rho(u^*,v^*))} \leq a.$$

However, since $\theta(\rho(u^*,v^*)) \geq 1$, the left side satisfies $(1-b)\theta(\rho(u^*,v^*)) \geq 1-b$, and the right side satisfies $\frac{2a}{1+\theta(\rho(u^*,v^*))} \leq \frac{2a}{1+1} = a$. Thus,

$$1-b \leq (1-b)\theta(\rho(u^*,v^*)) \leq a$$

Hence $1-b \leq a$,which implies $a+b \geq 1$, contradicting the hypothesis $a+b < 1$.
Therefore the assumptions $\rho(u^*,v^*) > 0$ is impossible, thus $\rho(u^*,v^*) = 0$ and the fixed point is unique. □

Corollary 2.2. *Let (U,ρ) be a complete metric space and let θ satisfy conditions $(\Theta_1)-(\Theta_3)$, and let the mapping $\mathfrak{T}: U \to U$ satisfy the θ-rational contraction*

$$\theta(\rho(\mathfrak{T}u,\mathfrak{T}v)) \leq \frac{a\,\theta(\rho(v,\mathfrak{T}v))\big[1+\theta(\rho(u,\mathfrak{T}u))\big]}{1+\theta(\rho(u,v))} + b\big(\theta(\rho(u,v))\big)^r, \qquad \text{for all } u,v \in U. \tag{5}$$

Then, by choosing $a = 0$, we obtain a generalized version of Jleli–Samet type inequality

$$\theta(\rho(\mathfrak{T}u,\mathfrak{T}v)) \leq b\big(\theta(\rho(u,v))\big)^r$$

and consequently $\mathfrak{T}$ has a unique fixed point $u^* \in U$. Moreover, for every $u_0 \in U$ the Picard iteration $u_{n+1} = \mathfrak{T}u_n$ converges to u^*.

Proof. By taking $a = 0$ in (5), the first term vanishes and we obtain exactly

$$\theta(\rho(\mathfrak{T}u,\mathfrak{T}v)) \leq b\big(\theta(\rho(u,v))\big)^r.$$

Since $b \in [0,1)$ and $r \in (0,1)$, the above inequality is a strict θ-contraction in the sense that the right side is a contraction factor less than 1 applied to $(\theta(\rho(u,v)))^r$. The hypotheses of the θ-rational contractions are therefore satisfied. □

Example 2.3 (Finite Space)**.** *Take $\theta(t) = 1+t^p$ with $p \in (0,1)$. Then:*

- *θ is continuous and non-decreasing;*
- *$\theta(0^+) = 1$ and $\theta(t) \to 1$ if and only if $t \to 0^+$;*
- $\dfrac{\theta(t)-1}{t^p} = \dfrac{t^p}{t^p} = 1.$

Let $U = \{0,1\}$ with the metric $\rho(0,1) = 1$. Choose $\theta(t) = 1+t^{1/2}$ so that $\theta(1) = 2$. Define the map $\mathfrak{T}: U \to U$ by

$$\mathfrak{T}(0) = \mathfrak{T}(1) = 0.$$

Now check the θ-rational contraction for $u = 0$, $v = 1$.

Left-hand side:

$$\theta(\rho(\mathfrak{T}u,\mathfrak{T}v)) = \theta(\rho(\mathfrak{T}0,\mathfrak{T}1)) = \theta(0) = \lim_{t\to 0^+} \theta(t) = 1.$$

Right-hand side: *Using* $\rho(v,\mathfrak{T}v)=\rho(1,0)=1$, $\rho(u,\mathfrak{T}u)=\rho(0,0)=0$, *and* $\rho(u,v)=1$, *we get*

$$RHS=\frac{a\,\theta(\rho(v,\mathfrak{T}v))[1+\theta(\rho(u,\mathfrak{T}u))]}{1+\theta(\rho(u,v))}+b\big(\theta(\rho(u,v))\big)^r$$

$$=\frac{a\,\theta(1)\big[1+\theta(0)\big]}{1+\theta(1)}+b\big(\theta(1)\big)^r=a\cdot\theta(1)+b\big(\theta(1)\big)^r.$$

Choose $a=0.4$, $b=0.4$, $r=\frac{1}{2}$. *Then* $a+b=0.8<1$. *Therefore,*

$RHS=0.4\cdot 2+0.4\cdot 2^{1/2}\approx 0.8+0.5657=1.3657>1.$

Thus,

$1\le RHS$

holds.

Example 2.4 (Continuous Example on an Interval). *Let* $U=[0,1]$ *together with the metric* $\rho(u,v)=|u-v|$. *Choose*

$\theta(t)=1+t^p, \qquad p\in(0,1).$

Consider the linear contraction $\mathfrak{T}(u)=\lambda u$ *with* $0\le\lambda<1$. *We want to choose parameters* a,b,r *so that for all* $u,v\in[0,1]$,

$$\theta(|\lambda u-\lambda v|)\le\frac{a\,\theta(|v-\lambda v|)\big[1+\theta(|u-\lambda u|)\big]}{1+\theta(|u-v|)}+b\big(\theta(|u-v|)\big)^r.$$

Since $\rho(u,v)\le 1$, *set* $t:=|u-v|\in[0,1]$. *Also,*

$|v-\lambda v|=(1-\lambda)v\in[0,1-\lambda], \qquad |u-\lambda u|\in[0,1-\lambda].$

Thus the inequality reduces to checking, for all $t\in[0,1]$,

$$1+(\lambda t)^p\le\frac{a\,\theta_{\max}(1+\theta_{\max})}{1+\theta(t)}+b\big(\theta(t)\big)^r,$$

where $\theta_{\max}:=\max_{s\in[0,1-\lambda]}\theta(s)=1+(1-\lambda)^p$, *since* $u,v\in[0,1]$.
The above inequality is equivalent to $\frac{1+(\lambda t)^p-a\,\theta_{\max}\frac{(1+\theta_{\max})}{1+\theta(t)}}{\big(\theta(t)\big)^r}\le b$.
Now, define the function

$$F(t):=\frac{1+(\lambda t)^p-a\,\theta_{\max}\frac{(1+\theta_{\max})}{1+\theta(t)}}{\big(\theta(t)\big)^r}.$$

we can find a,b,r *with* $a+b<1$ *such that* $\sup_{t\in[0,1]}F(t)\le b$,
Let $p=\frac{1}{2},\lambda=0.3,a=0.45,r=\frac{1}{2}$ *and* $b=0.545$. *Then,*

$$\sup F(t)\approx 0.5418\le 0.545.$$

Thus, the contraction inequality holds, and since the mapping $\mathfrak{T}$ *satisfies the* θ*-rational contraction inequality* (1) *for the chosen parameters* a,b,r *and the metric* $([0,1,|\cdot|])$ *is complete, it follows from Theorem* (2.1) *that* $\mathfrak{T}$ *possesses a unique fixed point in* $[0,1]$

Remark 2.5. *The* θ*-rational contraction introduced in Theorem 2.1 strictly generalizes several existing contraction conditions. In particular, when the rational term is omitted (that is, by setting* $a=0$*), as shown in Corollary 2.2, the contractive inequality reduces to a power-type* θ*-contraction, which includes the* θ*-contraction of Jleli and Samet as a special case. Moreover, the illustrative examples presented on both finite and continuous metric spaces further demonstrate the applicability and flexibility of the proposed framework.*

Theorem 2.6. *Let* (U,ρ) *be a complete metric space with* $\theta\in\Theta$. *Let* $\mathfrak{T},S:U\to U$ *be two mappings each satisfying the* θ*-rational contraction inequality*

$$\theta(\rho(Mu,Mv))\le\frac{a\,\theta(\rho(v,Mv))\big[1+\theta(\rho(u,Mu))\big]}{1+\theta(\rho(u,v))}+b\big(\theta(\rho(u,v))\big)^r,\qquad M=\mathfrak{T},S. \tag{6}$$

Assume that $\mathfrak{T}$ *and* S *commute, i.e.* $\mathfrak{T}S=S\mathfrak{T}$. *Then* $\mathfrak{T}$ *and* S *have a common fixed point* $u^*\in U$ *which is unique. Moreover, for every* $u_0\in U$ *the sequences* $\mathfrak{T}^n u_0$ *and* $S^n u_0$ *both converge to* u^*.

Proof. Since both $\mathfrak{T}$ and S satisfy the θ-rational contraction hypothesis with the same admissible function θ and constants a,b,r with $a+b<1$, the θ-rational contraction theorem applied to $\mathfrak{T}$ and to S separately yields that each map has a unique fixed point. Denote these by

$u_{\mathfrak{T}}\in U$ with $\mathfrak{T}(u_{\mathfrak{T}})=u_{\mathfrak{T}}, \qquad u_S\in U$ with $S(u_S)=u_S$.

Now use the commuting property to show the fixed points coincide. Compute

$S\big(\mathfrak{T}(u_S)\big)=\mathfrak{T}\big(S(u_S)\big)=\mathfrak{T}(u_S),$

so $\mathfrak{T}(u_S)$ is a fixed point of S. But S has a unique fixed point, namely u_S; hence

$\mathfrak{T}(u_S) = u_S.$

Thus u_S is fixed by both $\mathfrak{T}$ and S. By the uniqueness of $u_{\mathfrak{T}}$, we must have $u_{\mathfrak{T}} = u_S$. Set

$u^* := u_{\mathfrak{T}} = u_S.$

Thus u^* is the unique point fixed by both maps.
Next we show that both iterative sequences converge to this common fixed point u^*. Let $u_0 \in U$ be arbitrary. Applying the θ-rational contraction theorem to $\mathfrak{T}$ gives

$\mathfrak{T}^n u_0 \longrightarrow u_{\mathfrak{T}} = u^*.$

Similarly, applying it to S yields

$S^n u_0 \longrightarrow u_S = u^*.$

Since $u_{\mathfrak{T}} = u_S = u^*$, both iterative sequences converge to the same common fixed point u^*.
To prove uniqueness, suppose $v \in U$ is another common fixed point of $\mathfrak{T}$ and S. Then v is a fixed point of $\mathfrak{T}$, but $\mathfrak{T}$ has the unique fixed point $u_{\mathfrak{T}}$. Thus $v = u_{\mathfrak{T}} = u^*$. Hence the common fixed point is unique. □

3. Conclusion

We introduced the class of θ-rational contractions, unifying rational-type, power-type, and θ-contractive conditions in complete metric spaces. A general fixed point theorem was established, ensuring existence, uniqueness, and convergence of the Picard iteration, with a corollary showing that the framework generalizes and strengthens the Jleli-Samet θ-contraction. Illustrative examples and a common fixed point theorem for commuting maps demonstrated the versatility and applicability of the approach. Overall, the θ-rational framework provides a flexible, unified method for studying nonlinear contractive mappings and opens avenues for further generalizations and applications in analysis and optimization. Future research directions include extensions of the θ-rational framework to generalized metric spaces such as b-metric, rectangular metric, and partial metric spaces, as well as multivalued and cyclic mappings. Potential applications to nonlinear integral equations, fractional differential equations, and iterative algorithms in optimization may also be explored.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations integrals, *Fundamenta mathematicae* 3(1) (1922) 133-181.
[2] R. Kannan, Some results on fixed points. *Bull. Cal. Math. Soc.,* 60 (1968) 71-76.
[3] S. K. Chatterjea, Fixed-point theorems. *Dokladi na Bolgarskata Akademiya na Naukite,* 25(6) (1972) 727-730.
[4] G. E. Hardy, & T. D. Rogers, A generalization of a fixed point theorem of Reich. *Canadian Mathematical Bulletin,* 16(2) (1973) 201-206.
[5] M. Jleli, & B. Samet, A new generalization of the Banach contraction principle. *Journal of inequalities and applications,* 2014(1) (2014), 38.https://doi.org/10.1186/1029-242X-2014-38
[6] M. Jleli, E. Karapınar, & B. Samet, Further generalizations of the Banach contraction principle. *Journal of Inequalities and Applications,* 2014(1), (2014) 439. https://doi.org/10.1186/1029-242X-2014-439
[7] G. D. Güngör, & İ. Altun, Some fixed point results for α-admissible mappings on quasi metric space via θ-contractions. *Mathematical Sciences and Applications E-Notes,* 12(1), (2024) 12-19. https://doi.org/10.36753/mathenot.1300609
[8] H. Massit, M. Rossafi, Z. D. Mitrovic, A. Aloqaily, & N. Mlaiki, Some fixed point theorems for $(\theta-\phi)$-multivalued contraction mappings in rectangular b-metric spaces. *European Journal of Pure and Applied Mathematics,* 18(2), (2025) 6086-6086. https://doi.org/10.29020/nybg.ejpam.v18i2.6086
[9] U. Ishtiaq, U. Fahim, K. Ahmad, D. A. Kattan, & I. K. Argyros, Fixed point results for generalized-contraction. *Foundations* 3(3) (2023) 2673-9321. https://doi.org/10.3390/foundations3030028
[10] A. Ali, M. Arshad, A. Hussain, N. Hussain, & S. M. Alsulami, On new generalized θ_b-contractions and related fixed point theorems. *Journal of Inequalities and Applications,* 2022(1), (2022) 37.https://doi.org/10.1186/s13660-022-02770-8
[11] B. K. Dass, & S. Gupta, An extension of Banach contraction principle through rational expression. *Indian J. pure appl. Math,* 6(12) (1975) 1455-1458.
[12] D. S. Jaggi, Some unique fixed point theorems. *Indian J. Pure Appl. Math,* 8(2) (1977), 223-230.
[13] A. Hossain, F. A. Khan, & Q. H. Khan, A relation-theoretic metrical fixed point theorem for rational type contraction mapping with an application. *Axioms,* 10(4), (2021), 316.https://doi.org/10.3390/axioms10040316
[14] I. M. Batiha, B. Laouadi, I. H. Jebril, L. Benaoua, T. E. Oussaeif, & S. Alkhazaleh, Some new fixed point theorems of rational type contraction with an application to coupled fixed point theory. *Adv. Fixed Point Theory,* 14 (2024).https://doi.org/10.28919/afpt/8714
[15] M. Seddik, & H. Taieb, Some fixed point theorems of rational type contraction in b-metric spaces. *Moroccan Journal of Pure and Applied Analysis,* 7(3) (2021), 350-363.https://doi.org/10.2478/mjpaa-2021-0023
[16] L. Ciric, M. O. Olatinwo, D. Gopal, & G. Akinbo, Coupled fixed point theorems for mappings satisfying a contractive condition of rational type on a partially ordered metric space. *Adv. Fixed Point Theory,* 2(1), (2012) 1-8.
[17] M. Raji, L. Rathour, L. N. Mishra, & V. N. Mishra, Generalized rational type contraction and fixed point theorems in partially ordered metric spaces. *Journal of Advances in Applied & Computational Mathematics,* 10, (2023) 153-162.https://doi.org/10.15377/2409-5761.2023.10.13
[18] M. O. Olatinwo, & B. T. Ishola, Some fixed point theorems involving rational type contractive operators in complete metric spaces. *Surveys in Mathematics and its Applications,* 13, (2018) 107-117.