Complete monotonicity of a function involving the $p$-psi function and alternative proofs

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Abstract

In the paper, the authors prove that the function $x^{\alpha} \left[ \ln \frac{x^{p+1}}{p+1} - \psi_p(x) \right]$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$, where $p \in \mathbb{N}$ and $\psi_p(x)$ is the $p$-analogue of the classical psi function $\psi(x)$.

Keywords: completely monotonic function; necessary and sufficient condition; $p$-gamma function; $p$-psi function; inequality

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1. Introduction

Recall from [12, Chapter XIII], [16, Chapter 1] and [17, Chapter IV] that a function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and satisfies

$$0 \leq (-1)^n f^{(n)}(x) < \infty \quad (1.1)$$

for $x \in I$ and $n \geq 0$. The celebrated Bernstein-Widder’s Theorem (see [16, p. 3, Theorem 1.4] or [17, p. 161, Theorem 12b]) characterizes that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \, d\alpha(t), \quad (1.2)$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. This expresses that a completely monotonic function $f$ on $[0, \infty)$ is a Laplace transform of the measure $\alpha$. 
It is common knowledge that the classical Euler’s gamma function \( \Gamma(x) \) may be defined for \( x > 0 \) by
\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt.
\]
The logarithmic derivative of \( \Gamma(x) \), denoted by \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \), is called psi function or digamma function.

An alternative definition of the gamma function \( \Gamma(x) \) is
\[
\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x),
\]
where
\[
\Gamma_p(x) = \frac{p^p x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(x+1/1)\cdots(x+p/1)}
\]
for \( x > 0 \) and \( p \in \mathbb{N} \), the set of all positive integers. See [3, p. 250]. The \( p \)-analogue of the psi function \( \psi(x) \) is defined as the logarithmic derivative of the \( \Gamma_p \) function, that is,
\[
\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}.
\]

The function \( \psi_p \) has the following properties:

1. It has the following representations
\[
\psi_p(x) = \ln p - \sum_{k=0}^{p-1} \frac{1}{x+k} = \ln p - \int_0^\infty \frac{1 - e^{-(p+1)t}}{1 - e^{-t}} e^{-xt} \, dt.
\]

2. It is increasing on \((0, \infty)\) and \( \psi_p'(x) \) is completely monotonic on \((0, \infty)\).

The very right hand side of the formula \((1.6)\) corrects errors appeared in [8, p. 374, Lemma 5] and [10, p. 29, Lemma 2.3].

In [2, pp. 374–375, Theorem 1], it was proved that the function
\[
\theta_{\alpha}(x) = x^\alpha[\ln x - \psi(x)]
\]
is completely monotonic on \((0, \infty)\) if and only if \( \alpha \leq 1 \). For the history, background, applications and alternative proofs of this conclusion, please refer to [4, 13, p. 8, Section 1.6.6] and closely related references therein.

The aim of this paper is to generalize [2, pp. 374–375, Theorem 1] and [4, p. 105, Theorem 1] to the case of the \( p \)-analogue \( \psi_p(x) \) of the psi function \( \psi(x) \) as follows.

**Theorem 1.1.** The function
\[
\theta_{p,\alpha}(x) = x^\alpha \left[ \ln \frac{px}{x+p+1} - \psi_p(x) \right]
\]
for \( p \in \mathbb{N} \) is completely monotonic on \((0, \infty)\) if and only if \( \alpha \leq 1 \).

**Remark 1.1.** Letting \( p \to \infty \) in Theorem 1.1, we obtain [2, pp. 374–375, Theorem 1] and [4, p. 105, Theorem 1].

## 2. Proofs of Theorem 1.1

**First Proof.** From the identity \((1.6)\) and the integral expression
\[
\ln \frac{b}{a} = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt
\]
in [1, p. 230, 5.1.32], we obtain
\[
\theta_{p,1}(x) = x \int_0^\infty [1 - e^{-(p+1)t}] \varphi(t) e^{-xt} \, dt,
\]
where \( \varphi(t) \) is a suitable function.
where
\[ \varphi(t) = \frac{1}{1 - e^{-t}} - \frac{1}{t}. \] (2.3)

The function \( \varphi(t) \) is increasing on \((0, \infty)\) with
\[ \lim_{t \to 0^+} \varphi(t) = \frac{1}{2} \quad \text{and} \quad \lim_{t \to \infty} \varphi(t) = 1. \] (2.4)

See [5, 6, 7, 11, 14, 15, 18] and related references therein. Therefore, for \( x > 0 \) and \( n \in \mathbb{N} \), we have
\[
(-1)^n \theta_{p,1}^{(n)}(x) = x(-1)^n \frac{d^n}{dx^n} \int_0^{\infty} [1 - e^{-(p+1)t}] \varphi(t)e^{-xt} dt - (-1)^n \frac{d^{n-1}}{dx^{n-1}} \int_0^{\infty} [1 - e^{-(p+1)t}] \varphi(t)e^{-xt} dt
\]
\[= x \int_0^{\infty} t^n \varphi(t)[1 - e^{-(p+1)t}]e^{-xt} dt - n \int_0^{\infty} t^{n-1} \varphi(t)[1 - e^{-(p+1)t}]e^{-xt} dt
\]
\[= \int_0^{\infty} t^n \varphi(t)[1 - e^{-(p+1)t}]e^{-xt} dt - n \int_0^{\infty} t^{n-1} \varphi(t)[1 - e^{-(p+1)t}]e^{-xt} dt + \int_0^{\infty} t^{n-1} \varphi(t)(tx - n)e^{-xt} dt
\]
\[> \varphi \left( \frac{n}{x} \right) \int_0^{\infty} t^n \varphi(t)[1 - e^{-(p+1)t}]e^{-xt} dt - n \int_0^{\infty} t^{n-1} \varphi(t)[1 - e^{-(p+1)t}]e^{-xt} dt
\]
\[= \varphi \left( \frac{n}{x} \right) \int_0^{\infty} t^n e^{-xt} dt - n \int_0^{\infty} t^{n-1} e^{-xt} dt + n \int_0^{\infty} t^{n-1} e^{-(p+1)t} dt
\]
\[= \varphi \left( \frac{n}{x} \right) \left[ \int_0^{\infty} t^n e^{-xt} dt - \frac{n!}{x^{n+1}} - \frac{n!}{(x + p + 1)^{n+1}} + n \frac{(n - 1)!}{x^n} + n \frac{(n - 1)!}{(x + p + 1)^n} \right]
\]
\[= \varphi \left( \frac{n}{x} \right) \int_0^{\infty} t^n e^{-xt} dt - \frac{n!}{x^{n+1}} - \frac{n!}{(x + p + 1)^{n+1}} + n \frac{1}{x^n} + \frac{1}{(x + p + 1)^n}
\]
\[= \varphi \left( \frac{n}{x} \right) \int_0^{\infty} t^n e^{-xt} dt - \frac{n!}{x^{n+1}} - \frac{n!}{(x + p + 1)^{n+1}} + n \frac{1}{x^n} + \frac{1}{(x + p + 1)^n}
\]
\[> 0,
\]
where we used the formula
\[
\frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^{\infty} t^{n-1} e^{-xt} dt
\] (2.5)
for real numbers \( x > 0 \) and \( \omega > 0 \), see [1, p. 255, 6.1.1]. So we obtain that the function \( \theta_{p,1}(x) \) is completely monotonic on \((0, \infty)\).

Since
\[
(-1)^n [u(x)v(x)]^{(n)} = \sum_{i=0}^{n} \binom{n}{i} \left[ (-1)^i u^{(i)}(x) \right] \left[ (-1)^{n-i} v^{(n-i)}(x) \right],
\]
the product of any two completely monotonic function is also completely monotonic on their common domain. On the other hand, the function \( x^{\alpha-1} \) for \( \alpha < 1 \) is clearly completely monotonic on \((0, \infty)\). Consequently the function
\[ \theta_{p,\alpha}(x) = x^{\alpha-1} \theta_{p,1}(x) \]
for \( \alpha \leq 1 \) is completely monotonic on \((0, \infty)\).

Conversely, if \( \theta_{p,\alpha}(x) \) is completely monotonic on \((0, \infty)\), then
\[
\frac{d \theta_{p,\alpha}(x)}{dx} = x^{\alpha-1} \left\{ \alpha \left[ \ln \frac{px}{x + p + 1} - \psi_p(x) \right] + \frac{p + 1}{x + p + 1} - x \psi'_p(x) \right\} \leq 0
\]
for $x > 0$, equivalently,

$$\alpha \leq \frac{x\psi_p'(x) - \frac{p+1}{x+p+1}}{\ln \frac{px}{x+p+1} - \psi_p(x)}.$$ 

Employing L'Hôpital's rule and (1.6) results in

$$\lim_{x \to \infty} \frac{x\psi_p'(x) - \frac{p+1}{x+p+1}}{\ln \frac{px}{x+p+1} - \psi_p(x)} = \lim_{x \to \infty} \frac{x\psi_p'(x) + \psi_p'(x) + \frac{p+1}{(x+p+1)^2}}{\frac{1}{x} - \frac{1}{x+p+1} - \sum_{k=0}^{p} \frac{1}{(x+k)^2}} = 1,$$

so it is necessary that $\alpha \leq 1$. The proof is complete. \hfill \Box

**Second Proof.** From (2.2) and by integration by part lead to

$$\theta_{p,1}(x) = -\int_0^\infty \left[1 - e^{-(p+1)t}\right] \varphi(t) \frac{d}{dt} e^{-xt} \ dt - \left\{ \int_0^\infty \left[1 - e^{-(p+1)t}\right] \varphi(t) \right\} e^{-xt} \bigg|_{t=0}^{t=\infty}$$

$$= \int_0^\infty \left\{ [1 - e^{-(p+1)t}] \varphi(t) + (p+1)e^{-(p+1)t} \varphi(t) \right\} e^{-xt} \ dt.$$ 

Therefore, for showing that the function $\theta_{p,1}(x)$ is completely monotonic on $(0, \infty)$ for all $p \in \mathbb{N}$, it suffices to prove that the function

$$[1 - e^{-(p+1)t}] \varphi'(t) + (p+1)e^{-(p+1)t} \varphi(t)$$

is positive. Since the function $\varphi(t)$ is increasing on $(0, \infty)$, the derivative $\varphi'(t)$ is positive on $(0, \infty)$. Further considering the limits in (2.4), the positivity of $\varphi(t)$ follows. As a result, the function (2.6) is positive.

The rest of the proof is the same as the first proof. \hfill \Box

**Remark 2.1.** This paper is a slightly modified version of the preprint [9].

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