Majorization problems for $p$–valently meromorphic functions of complex order involving certain integral operator

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Abstract

The main object of this paper is to investigate the problem of majorization of certain class of meromorphic $p$-valent functions of complex order involving certain integral operator. Moreover we point out some new or known consequences of our main result.

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1 Introduction

Let $f$ and $g$ are analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Due to MacGregor [8], (also see [7]) we say that $f$ is majorized by $g$ in $\Delta$ and we write

\[ f(z) \ll g(z), \quad (z \in \Delta) \quad (1) \]

if there exists a function $\phi$, analytic in $\Delta$, such that

\[ |\phi(z)| < 1 \quad \text{and} \quad f(z) = \phi(z)g(z), \quad z \in \Delta. \quad (2) \]

It may be noted here that (1) is closely related to the concept of quasi-subordination between analytic functions. Also we say that $f$ is subordinate to $g$ denoted by $f \prec g$ (see [9]), if there exists a Schwarz function $\omega$ which is analytic in $\Delta$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \Delta$, such that

\[ f(z) = g(\omega(z)), \quad z \in \Delta. \]

We denote this subordination by $f \prec g$. Furthermore, if the function $g$ is univalent in $\Delta$, we have

\[ f \prec g \iff f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta). \]

Denote by $S^*(\gamma)$ and $C(\gamma)$ the class of starlike and convex functions of complex order $\gamma (\gamma \in \mathbb{C} \setminus \{0\})$, satisfying the following conditions

\[ \frac{f(z)}{z} \neq 0 \quad \text{and} \quad \Re \left( 1 + \frac{1}{\gamma} \left[ \frac{zf'(z)}{f(z)} - 1 \right] \right) > 0 \]
and

\[ f'(z) \neq 0 \quad \text{and} \quad \Re \left( 1 + \frac{1}{\gamma} \left[ \frac{zf''(z)}{f'(z)} \right] \right) > 0, \ (z \in \Delta) \]

respectively. Further,

\[ S^*((1 - \delta)\cos \lambda e^{-i\lambda}) = S^*(\delta, \lambda), \ |\lambda| < \frac{\pi}{2}; \ 0 \leq \delta \leq 1 \]

the class of \( \lambda \)-spirallike function of order \( \delta \) investigated by Libera [4] and

\[ S^*(\cos \lambda e^{-i\lambda}) = S^*(\lambda), \ |\lambda| < \frac{\pi}{2}, \]

the class of spiral-like functions introduced by Spacek [10] (also see [11]).

A majorization problem for the class of analytic starlike functions have been investigated by MacGregor [8] and Altintas et al. [1]. Recently Goyal and Goswami [3] extended these results for the class of meromorphic functions making use of certain integral operator.

Let \( \Sigma_p \) be the class of \( p \)-valently meromorphic functions which are analytic and univalent in the punctured unit disk

\[ \Delta^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = \Delta \setminus \{ 0 \} \]

of the form

\[ f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p}. \quad (3) \]

with a simple pole at the origin.

Due to Aqlan et al. [2] (see [5]), we recall the integral operator \( J_{\beta,p}^\alpha \) for meromorphic functions \( f \in \Sigma_p \) as given below,

\[ J_{\beta,p}^\alpha : \Sigma_p \rightarrow \Sigma_p \]

\[ J_{\beta,p}^\alpha f(z) = \left( \frac{\alpha + \beta - 1}{\beta - 1} \right) \frac{1}{z^{\beta+p}} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta+p-1} f(t) \ dt \quad (4) \]

\[ J_{\beta,p}^\alpha f(z) = \begin{cases} f(z) & \alpha = 0, \beta > -1, p \in \mathbb{N}, f \in \Sigma_p \\ \frac{1}{z^p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(\alpha+\beta)} a_{n-p} z^{n-p}, & \alpha > 0, \beta > -1, p \in \mathbb{N}, f \in \Sigma_p. \end{cases} \quad (5) \]

The following relation for \( J_{\beta,p}^\alpha f(z) \) can be obtained by simple calculation,

\[ z(J_{\beta,p}^\alpha f(z))' = (\alpha + \beta - 1)J_{\beta,p}^{\alpha-1} f(z) - (\alpha + \beta + p - 1)J_{\beta,p}^\alpha f(z). \quad (6) \]

Using (6), the below recurrence relation for \( J_{\beta,p}^\alpha f(z) \) can be obtained easily,

\[ z(J_{\beta,p}^\alpha f(z))^{(q+1)} = (\alpha + \beta - 1)(J_{\beta,p}^{\alpha-1} f(z))^{(q)} - (\alpha + \beta + p + q - 1)(J_{\beta,p}^\alpha f(z))^{(q)}. \quad (7) \]

In the present paper we investigate a majorization problem for the class of \( p \)-valently meromorphic starlike functions of complex order associated with the generalized integral operator due to Aqlan [2] and Murugasundaramoorthy and Magesh [6].

**Definition 1.1.** A function \( f(z) \in \Sigma_p \) is said to in the class \( M_{\alpha,\beta}^{p,q}(\gamma,A,B) \) of meromorphic functions of complex order \( \gamma \neq 0 \) in \( \Delta^* \) if and only if

\[ 1 - \frac{1}{\gamma} \left[ \frac{z(J_{\beta,p}^\alpha f(z))^{(q+1)}}{(J_{\beta,p}^\alpha f(z))^{(q)}} \right] + p + q < \frac{1 + Az}{1 + Bz}, \quad (8) \]

where \( z \in \Delta^*, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \}, \beta > -1, \alpha > 0, \gamma \in \mathbb{C} \setminus \{ 0 \} \) and \(-1 \leq B < A \leq 1\).
For simplicity, we put
\[ \mathcal{M}_{\alpha,\beta}^{p,q}(\gamma, A, B) = \mathcal{M}_{\alpha,\beta}^{p,q}(\gamma), \]
where \( \mathcal{M}_{\alpha,\beta}^{p,q}(\gamma) \) denote the class of functions \( f \in \Sigma_p \) satisfying the following inequality:
\[ \Re \left( 1 - \frac{1}{\gamma} \left[ \frac{z(\mathcal{J}_{\beta,p}^\alpha f(z))(q+1)}{(\mathcal{J}_{\beta,p}^\alpha f(z))^q} + p + q \right] \right) > 0 \]  (9)
where \( z \in \Delta^*, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta > -1, \alpha > 0, \gamma \in \mathbb{C} \setminus \{0\} \).

**Example 1.2.** Putting \( \gamma = (p - \delta)\cos \lambda e^{-i\lambda}, |\lambda| < \frac{\pi}{2}; 0 \leq \delta < p \) the class
\[ \mathcal{M}_{\alpha,\beta}^{p,q}(\gamma) = \mathcal{M}_{\alpha,\beta}^{p,q}((p - \delta)\cos \lambda e^{-i\lambda}) \equiv \mathcal{M}_{\alpha,\beta}^{p,q}(\delta, \lambda) \]
called the generalized class of \( \lambda \)-spiral-like functions of order \( \delta(0 \leq \delta < p) \) if
\[ \Re \left( e^{i\lambda} \left[ \frac{z(\mathcal{J}_{\beta,p}^\alpha f(z))(q+1)}{(\mathcal{J}_{\beta,p}^\alpha f(z))^q} + q \right] \right) < -\delta \cos \lambda \]  (10)
where \( z \in \Delta^*, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta > -1, \alpha > 0, \gamma \in \mathbb{C} \setminus \{0\} \).

**Example 1.3.** Putting \( \gamma = (p - \delta); 0 \leq \delta < p \) the class \( \mathcal{M}_{\alpha,\beta}^{p,q}(\delta) \equiv \mathcal{M}_{\alpha,\beta}^{p,q}(\delta) \), the generalized class of \( p \)-valently meromorphic starlike functions of order \( \delta(0 \leq \delta < p) \) if
\[ \Re \left( \frac{z(\mathcal{J}_{\beta,p}^\alpha f(z))(q+1)}{(\mathcal{J}_{\beta,p}^\alpha f(z))^q} + q \right) < -\delta \]  (11)
where \( z \in \Delta^*, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta > -1, \alpha > 0, \gamma \in \mathbb{C} \setminus \{0\} \).

**Remark 1.4.** By taking \( q = 0 \) in Example 1.3, \( \mathcal{M}_{\alpha,\beta}^{p,q}(\delta) \equiv \mathcal{M}_{\alpha,\beta}^{p,q}(\delta) \) the class of \( p \)-valently meromorphic starlike functions of order \( \delta(0 \leq \delta < p) \) if
\[ \Re \left( \frac{z(\mathcal{J}_{\beta,p}^\alpha f(z))'}{(\mathcal{J}_{\beta,p}^\alpha f(z))} \right) < -\delta \]
where \( z \in \Delta^*, \beta > -1, \alpha > 0, \gamma \in \mathbb{C} \setminus \{0\} \).

### 2 Majorization problem for the class \( \mathcal{M}_{\alpha,\beta}^{p,q}(\gamma, A, B) \)

**Theorem 2.1.** Let the function \( f \in \Sigma_p \) and \( g \in \mathcal{M}_{\alpha,\beta}^{p,q}(\gamma, A, B) \) if \( (\mathcal{J}_{\beta,p}^\alpha f(z))^q \) is majorized by \( (\mathcal{J}_{\beta,p}^\alpha g(z))^q \) in \( \Delta^* \) then
\[ |(\mathcal{J}_{\beta,p}^\alpha - 1 f(z))^q| \leq |(\mathcal{J}_{\beta,p}^\alpha - 1 g(z))^q|, \quad |z| \leq r_1, \]
where \( r_1 = r_1(A, B, \alpha, \beta, \gamma, \rho) \) is the smallest positive root of the equation
\[ |(\alpha + \beta - 1)B - \gamma(A - B)| r^3 - \{(\alpha + \beta - 1) + 2\rho|B| \} r^2 - \{(\alpha + \beta - 1)B - \gamma(A - B)| + 2\rho \} r + (\alpha + \beta - 1) = 0, \]  (13)
where \( z \in \Delta^*, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta > -1, \alpha > 0, \gamma \in \mathbb{C} \setminus \{0\} \) and \( -1 \leq B < A \leq 1 \).

**Proof.** Since \( g(z) \in \mathcal{M}_{\alpha,\beta}^{p,q}(\gamma, A, B) \), we readily obtain from (8) that, if
\[ 1 - \frac{1}{\gamma} \left[ \frac{z(\mathcal{J}_{\beta,p}^\alpha g(z))(q+1)}{(\mathcal{J}_{\beta,p}^\alpha g(z))^q} + p + q \right] = \frac{1 + Aw(z)}{1 + Bw(z)} \]  (14)
where \( w \) denotes the well known class of bounded analytic functions in \( \Delta \) and
\[ w(0) = 0 \quad \text{and} \quad |w(z)| \leq |z|, \quad (z \in \Delta). \]  (15)
From (14), we get
\[
\frac{z(J_{\beta,p}^\alpha g(z))^{(q+1)}}{(J_{\beta,p}^\alpha g(z))^{(q)}} = -\frac{(p+q) + [(p+q)B + \gamma(A-B)]w(z)}{1+Bw(z)}
\] (16)

Using (7) in the above equation, we get,
\[
(J_{\beta,p}^\alpha g(z))^{(q)} = \frac{(\alpha + \beta - 1)[1 + Bw(z)]}{(\alpha + \beta - 1) + [(\alpha + \beta - 1)B - \gamma(A-B)] w(z)} (J_{\beta,p}^{\alpha-1} g(z))^{(q)}.
\] (17)

Hence, by making use of (15), we get,
\[
|J_{\beta,p}^{\alpha-1} g(z)| \leq \frac{(\alpha + \beta - 1)[1 + |B||z|]}{(\alpha + \beta - 1) - [(\alpha + \beta - 1)B - \gamma(A-B)] |z|} |J_{\beta,p}^{\alpha-1} g(z)|^{(q)}.
\] (18)

Since \((J_{\beta,p}^\alpha f(z))^{(q)}\) is majorized by \((J_{\beta,p}^\alpha g(z))^{(q)}\) in \(\Delta^*\) from (2), we have
\[
(J_{\beta,p}^\alpha f(z))^{(q)} = \phi(z)(J_{\beta,p}^\alpha g(z))^{(q)}
\]

Differentiating the above equation w.r.t \(z\) and multiplying by \(z\), we have,
\[
z(J_{\beta,p}^\alpha f(z))^{(q+1)} = z\phi'(z)(J_{\beta,p}^\alpha g(z))^{(q)} + z\phi(z)(J_{\beta,p}^\alpha g(z))^{(q+1)}.
\]

By using (7), we get,
\[
(J_{\beta,p}^{\alpha-1} f(z))^{(q)} = \frac{z}{\alpha + \beta - 1} \phi'(z)(J_{\beta,p}^\alpha g(z))^{(q)} + \phi(z)(J_{\beta,p}^{\alpha-1} g(z))^{(q)}.
\] (19)

Noting that the Schwarz function \(\phi(z)\) satisfies
\[
|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2}
\] (20)

and using (18) and (20) in (19) we have
\[
|J_{\beta,p}^{\alpha-1} f(z)|^{(q)} \leq \left( |\phi(z)| + \frac{1-|\phi(z)|^2}{1-|z|^2} \cdot \frac{|z| (1+|B||z|)}{(\alpha + \beta - 1) - [(\alpha + \beta - 1)B - \gamma(A-B)] |z|} \right) |J_{\beta,p}^{\alpha-1} g(z)|^{(q)}
\]

which upon setting
\[
|z| = r \quad \text{and} \quad |\phi(z)| = \rho, \quad (0 \leq \rho \leq 1)
\]

leads us to the inequality
\[
|J_{\beta,p}^{\alpha-1} f(z)|^{(q)} \leq \frac{\theta(\rho)}{(1 - r^2)((\alpha + \beta - 1) - [(\alpha + \beta - 1)B - \gamma(A-B)] |r|)} |J_{\beta,p}^{\alpha-1} g(z)|^{(q)},
\] (21)

where
\[
\theta(\rho) = \rho(1 - r^2)((\alpha + \beta - 1) - [(\alpha + \beta - 1)B - \gamma(A-B)] |r|) + (1 - \rho^2)(1 + |B| |r|) r
\]
takes its maximum value at \(\rho = 1\). Furthermore, if \(0 \leq \sigma \leq r_1\), the function \(\varphi(\rho)\) defined by
\[
\varphi(\rho) = \rho (1 - \sigma^2)((\alpha + \beta - 1) - [(\alpha + \beta - 1)B - \gamma(A-B)] |\sigma|) + (1 - \rho^2)(1 + |B| |\sigma|) \sigma
\]
is an increasing function on \((0 \leq \rho \leq 1)\) so that
\[
\varphi(\rho) \leq \varphi(1) = (1 - \sigma^2)((\alpha + \beta - 1) - [(\alpha + \beta - 1)B - \gamma(A-B)] |\sigma|).
\] (22)

Therefore, from this fact, (21) gives the inequality (12). \(\Box\)
3 Corollaries and Concluding Remarks

By taking $A = 1; B = -1$ and $\rho = 1$ in Theorem 2.1, we state the following corollary without proof.

**Corollary 3.1.** Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}_{\alpha, \beta}^p(\gamma)$ if $(\mathcal{J}_{\beta, p}^\alpha f(z))^{(q)}$ is majorized by $(\mathcal{J}_{\beta, p}^\alpha g(z))^{(q)}$ in $\Delta^*$ then

$$|(\mathcal{J}_{\beta, p}^\alpha f(z))^{(q)}| \leq |(\mathcal{J}_{\beta, p}^\alpha g(z))^{(q)}|, \ |z| \leq r_2,$$

where $r_2 = r_2(\alpha, \beta, \gamma)$ is the smallest positive root of the equation

$$\{[(\alpha + \beta - 1) + 2\gamma]r^3 - (\alpha + \beta + 1)r^2 - |(\alpha + \beta - 1) + 2\gamma| r + (\alpha + \beta - 1) = 0, \text{ given by}
\]

$$r_2 = \frac{L_1 - \sqrt{L_1^2 - 4|\alpha + \beta - 1 + 2\gamma|(\alpha + \beta - 1)}}{2|\alpha + \beta - 1 + 2\gamma|}
$$

and $L_1 = \alpha + \beta + 1 + |\alpha + \beta - 1 + 2\gamma|$. By setting $\alpha = 1$ in Corollary 3.1, we state the following corollary.

**Corollary 3.2.** Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}_{\alpha, \beta}^p(\gamma)$ if $(\mathcal{J}_{\beta, p}^1 f(z))^{(q)}$ is majorized by $(\mathcal{J}_{\beta, p}^1 g(z))^{(q)}$ in $\Delta^*$ then

$$|(f(z))^{(q)}| \leq |(g(z))^{(q)}|, \ |z| \leq r_3,$$

where $r_3 = r_3(1, \beta, \gamma)$ is the smallest positive root of the equation

$$|\beta + 2\gamma| r^3 - (\beta + 2) r^2 - |(\beta + 2\gamma) + 2| r + \beta = 0, \text{ given by}
\]

$$r_3 = \frac{L_2 - \sqrt{L_2^2 - 4|\beta + 2\gamma| |\beta + 2\gamma|}}{2|\beta + 2\gamma|}
$$

and $L_2 = \beta + 2 + |\beta + 2\gamma|$. By setting $\alpha = 1, \beta = 1$ and $\gamma = p - \delta$ in Corollary 3.1, we state the following corollary.

**Corollary 3.3.** Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}_{\alpha, \beta}^p(\delta)$ if $(\mathcal{J}_{\beta, p}^1 f(z))^{(q)}$ is majorized by $(\mathcal{J}_{\beta, p}^1 g(z))^{(q)}$ in $\Delta^*$ then

$$|(f(z))^{(q)}| \leq |(g(z))^{(q)}|, \ |z| \leq r_4,$$

where $r_4 = r_4(1, 1, (p - \delta)1)$ is the smallest positive root of the equation

$$|1 + 2(p - \delta)| r^3 - 3r^2 - |1 + 2(p - \delta)| + 2| + 2| r + 1 = 0, \text{ given by}
\]

$$r_4 = \frac{L_3 - \sqrt{L_3^2 - 4|1 + 2(p - \delta)| |1 + 2(p - \delta)|}}{2|1 + 2(p - \delta)|}
$$

and $L_3 = 3 + |1 + 2(p - \delta)|$. **Remark 3.4.** By taking $p = 1$ and $q = 0$, Corollary 3.3 yields results of Goyal and Gosami[3].

By taking $\gamma = (p - \delta)\cos \lambda e^{-i\lambda}$ ($|\lambda| < \frac{\pi}{2}, 0 \leq \delta < p$), in Corollary 3.1, we state the following corollary without proof.

**Corollary 3.5.** Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}_{\alpha, \beta}^p(\alpha, \lambda)$ if $(\mathcal{J}_{\beta, p}^\alpha f(z))^{(q)}$ is majorized by $(\mathcal{J}_{\beta, p}^\alpha g(z))^{(q)}$ in $\Delta^*$ then

$$|(\mathcal{J}_{\beta, p}^\alpha f(z))^{(q)}| \leq |\mathcal{J}_{\beta, p}^\alpha g(z))^{(q)}|, \ |z| \leq r,$$

where $r = r(T, \lambda)$ is given by

$$r = \frac{T - \sqrt{T^2 - 4|\alpha + \beta - 1 + 2(p - \delta)\cos \lambda e^{-i\lambda}|(\alpha + \beta - 1)}}{2|\alpha + \beta - 1 + 2(p - \delta)\cos \lambda e^{-i\lambda}|}
$$

and

$$T = (\alpha + \beta + 1) + |1 + 2(p - \delta)\cos \lambda e^{-i\lambda}|.$$

**Concluding Remarks:** Further specializing the parameters $\alpha, \beta$ one can define the various other interesting subclasses of $\Sigma_p$ involving the various integral operators and the corresponding corollaries as mentioned above can be derived easily.
References


