Discrete prey-predator model with Beddington-DeAngelis functional response: simple vs. complex dynamics

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Abstract

In this paper the dynamics of a discrete-time prey-predator system is investigated in the closed first quadrant $\mathbb{R}^2_+$. The existence and stability of fixed points are analyzed algebraically. The conditions of existence for flip bifurcation is derived by using center manifold theorem and bifurcation theory. Numerical simulations not only illustrate our results but also exhibit complex dynamical behaviors of the model, such as the periodic-doubling bifurcation in periods 2, 4 and 8 and quasi-periodic orbits and chaotic sets.

Keywords: Discrete Model, Beddington-Deangelis Functional Response, Stability, Flip Bifurcation, Center Manifold Theorem, Numerical Simulation

1. Introduction

It is well known the Lotka-Voltera prey-predator model is one of the fundamental population models, a predator-predator interaction has been described firstly by two pioneers Lotka [18] and Voltera [22] in two independent works. After them, more realistic prey-predator model were introduced by Holling suggesting three types of functional responses for different species to model the phenomena of predation [9]. Predator-prey models have already received much attention from many authors. For example, the stability, and the existence of periodic solutions of the predator prey models are studied in [16], [19], [20], [23]. The dynamics of a prey-predator differential equations model is studied by many authors see [6], [7], [15]. Another possible way to understand a prey-predator model is by using discrete models. Actually these models are more reasonable than the continuous time models when populations have non-overlapping generations see [14]. Discrete-time models can give rise to more efficient computational models for numerical simulations and it exhibits more plentiful dynamical behaviors than a continuous-time model of the same type. For continuous-time predator-prey models, many authors have chosen delay as the bifurcation parameter to discuss the Hopf bifurcation in [13, 21]. However, there are few articles discussing the dynamical behaviors of predator-prey models, which include bifurcations and chaos phenomena for the discrete-time models. Liu and Xiao [17], Jing and Yang [11], He and Lai [8], Hu et al. [10] obtained the flip bifurcation by using the center manifold theorem and bifurcation theory. But Agiza et al. [1] and Celik et al. [3] only showed the flip bifurcation and Hopf bifurcation by using numerical simulations. A functional response is called of Beddington-DeAngelis type if it takes the form $P(x, y) = \frac{kx}{(1+by+ax)}$. This type of functional response was introduced by Beddington [2] and DeAngelis et al. [4]. The term $by$ measures the mutual interference between predators. In this paper we consider the following continuous-time prey-predator model with Beddington-DeAngelis type of functional response described by differential equations

$$
\begin{align*}
\frac{dx_1}{dt} &= r x_1 \left(1 - \frac{x_1}{K}\right) - \frac{m x_1 x_2}{B x_2 + h x_1 + 1} \\
\frac{dx_2}{dt} &= c x_1 m x_2 x_2 - d x_2
\end{align*}
$$

The model parameters $r, K, m, B, h, c, d$ and $d$ are assuming only positive values. The prey $x_1$ grows with intrinsic growth rate $r$ and carrying capacity $K$ in the absence of predation. The predator $x_2$ consumes the prey $x_1$ with functional
response Beddington-DeAngelis type. The constant $c_1$ is conversion rates of prey to predator, $d_1$ is predator death rate for species $x_2$. by using the following transformation $t = rT, x_1 = Kx, x_2 = \frac{r}{m}$ we get the following system
\begin{align}
\frac{dx}{dt} &= x(1-x) - \frac{xy}{1+ax+by} \\
\frac{dy}{dt} &= \frac{cx}{1+ax+by} - ey \tag{2}
\end{align}
Where $a = \frac{hr}{m}, b = \frac{br}{r}, c = \frac{c_1mK}{r}$, and $e = \frac{d_1}{r}$. Applying the forward Euler scheme to system (2), we obtain the discrete-time system as follows:
\begin{align}
x &\rightarrow x + dx \left[ 1 - x - \frac{y}{1+ax+by} \right] \\
y &\rightarrow y + dy \left[ \frac{cx}{1+ax+by} - e \right] \tag{3}
\end{align}
Where $d$ is the step size. The main goal of this paper is to investigate this version as a discrete-time dynamical system in the interior of the first quadrant by using bifurcation theory and center manifold theory. The organization of this paper is as follows. In the second section we discuss the existence and local stability of fixed points in model (3). In the third section study the numerical simulations, which not only illustrate our results with theoretical analysis but also exhibit complex dynamical behaviors such as the cascade periodic-doubling bifurcation in periods 2, 4 and 8 and quasi-periodic orbits and chaotic sets. In section four we give the discussion for our work.

2. The existence and stability of fixed points

In this section, we consider the discrete-time model (3) in the closed first quadrant $\mathbb{R}^2_+$ on the xy-plane. We first discuss the existence of the fixed points for (3), and then study the stability of the fixed point by the eigenvalues for the variational matrix of (3) at the fixed point.

To determine the fixed points of (3) we have to solve the non-linear system given by
\begin{align}
x &\rightarrow x + dx \left[ 1 - x - \frac{y}{1+ax+by} \right] \\
y &\rightarrow y + dy \left[ \frac{cx}{1+ax+by} - e \right] \tag{4}
\end{align}
By composition of the above algebraic system we obtain the following results:

Lemma 2.1 System (3):

1) Has two fixed points; for all parameter values,
   a) $E_0(0,0)$ is origin?
   b) $E_1(1,0)$ is the axial fixed point.
2) Has the interior fixed point $E_2(x^*, y^*)$ where
   \[ y^* = \frac{(c-ae)x^*-e}{be} \text{ And } 0 < \frac{e}{c-ae} < x^* < 1. \]

Now we study the stability of these fixed points. Note that the local stability of the fixed point $(x, y)$ is determined by the modules of eigenvalues of the characteristic equation at the fixed point.

The Jacobian matrix of (3) evaluated at the point $(x, y)$ is given by
\[
J(x, y) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}
\]
Where
\[
j_{11} = 1 + dx \left( -1 + \frac{ay}{(1+ax+by)^2} + d \left( 1 - x - \frac{y}{1+ax+by} \right) \right); \\
j_{12} = -dx(1+ax) \left( \frac{1}{1+ax+by} \right); \\
j_{21} = \frac{dcy(1+by)}{(1+ax+by)^2}; \\
j_{22} = 1 - \left( \frac{bcxy}{(1+ax+by)^2} \right) \left( \frac{cx}{1+ax+by} - e \right) d
\]
And the characteristic equation of the Jacobian matrix $J(x, y)$ can be written as
\[ \lambda^2 + P(x, y)\lambda + Q(x, y) = 0 \tag{5} \]
Where
\[ P(x, y) = -(j_{11} + j_{22}) \]
\[ Q(x, y) = j_{11}j_{22} - j_{12}j_{21}. \]
In order to discuss the stability of the fixed points of (3), we also need the following Lemma.

Lemma 2.2 Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0, \lambda_1, \lambda_2$ are two roots of $\lambda(\lambda) = 0$. Then:

1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$;
2) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;
and if and only if 

If and are complex and if and only if and .

Let and be two roots of (5). We recall some definitions of topological types for a fixed point (x, y).

**Definition 2.3:** A fixed point (x, y) is called a sink if and only if , a source if and only if , and a saddle if and only if or .

A fixed point is called a sink if and if and only if .

Proposition 2.4: The fixed point is a source if , is a saddle if , and is non-hyperbolic if either or .

At the fixed point the roots of equation (5) are with , so is not one with model and the second eigenvalue when . This periodic doubling bifurcation may occur where parameters vary in the neighborhood .

The next proposition shows the local dynamics of fixed point from Lemma (2.2).

Proposition 2.5: There are at least four different topological types of for all permissible values of parameters

1) is a sink if , and is a source if or , and is a saddle if or , and is non-hyperbolic if or .

From the above proposition we obtain, for a fixed point if (a, b, c, d, e) ∈ R

Where

then one of the eigenvalues of (5) is -1 and the other is neither 1 nor -1. Therefore, there may be flip bifurcation of a fixed point , if d varies in the small neighborhood of .

Proposition 2.6: Let be the positive fixed point of (3). Then

1) is a sink if one of the following conditions holds?

a) And ;

b) And ;

2) is a source if one of the following conditions holds?

a) And ;

b) And ;

3) is a saddle if one of the following conditions holds?

a) And ;

b) And ;

4) is non-hyperbolic if one of the following conditions holds

a) And ;

b) And ;

Where

From Lemma (2.2), we can see that one of the eigenvalues of is -1 and the other is neither 1 nor -1 if the term (4a) in Proposition (2.6) holds. Therefore there may be flip bifurcation of if d varies in the small neighborhood of .

Where

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Where
Also when the term (4b) in proposition (2.6) holds, we can obtain that the eigenvalues of $E_2$ are pair of conjugate complex numbers with $|\lambda_1| = 1, |\lambda_2| = 1$ and hence the conditions in term (4b) of proposition (2.6) can be written as 

$$C = \left\{ (a, b, c, d, e) \bigg| d = \frac{N^*}{M^*}, P^* < 0, a, b, c, d, e > 0 \right\}$$

If the parameter $d$ varies in the small neighborhood of $C$, then the Hopf bifurcation will appear.

3. Numerical simulations

In this section, we give the bifurcation diagrams, phase portraits of model (3) to confirm the above theoretical analysis and show the dynamical behaviors by using numerical simulations. The bifurcation parameters are considered in the following three cases:

**Case 1:** Choosing $a = 0.3$, $b = 0.5$, $c = 0.1$, and $e = 0.2$, initial value $(x_0, y_0) = (0.5, 0.6)$ and $d \in [0, 2.96]$. We see that model (3) has a fixed point $E_1(1, 0)$ and $(a, b, c, d) \in R$. Figure 1 show the correctness of proposition (2.5). From Figure 1 we see that the fixed point is stable for $d < 2$, and loses its stability when $d = 2$. Further, when $d > 2$ there is the period-doubling bifurcation. Moreover a chaotic set is emerged with increasing of $d$.

![Fig. 1: The Flip Bifurcation of X (N) With A=0.3, B=0.5, C=0.1, And E=0.2,d \in [0, 2.96] And Initial(x_0,y_0) = (0.5, 0.6).](image)

**Case 2:** Choosing $a = 1.3$, $b = 3$, $c = 2$, and $e = 0.4$, initial value $(x_0, y_0) = (0.4, 0.6)$ and $d \in [1.5, 3.5]$ and we only vary parameter $d$ to see the variation of dynamics behaviors about model (3), on the basis of proposition (2.6), we know that the system has only one positive fixed point. After calculation for the positive fixed point of map (3), the flip bifurcation emerges from the fixed point $(5/6, 25/36)$ at $d = 2.6855$ and $(a, b, c, d, e) = (1.3, 3, 2, 2.6855, 0.4) \in B^*_1$. It shows the correctness of proposition (2.6). From Figure 2 we see that the fixed point is stable for $d < 2.6855$, and loses its stability when $d = 2.6855$. We also observe that there is a cascade of period-doubling bifurcation.

![Fig. 2: The Bifurcation Diagram of Model (3) In The (D, X)-Plane For A = 1.3, B=3, C=2, And E = 0.4,d \in [1.5, 3.5]And Initial(x_0,y_0) = (0.4, 0.6).](image)

**Case 3:** Choosing $a=1$, $b = 0.5$, $c = 4$, and $e = 9/10$, initial value $(x_0, y_0) = (0.4, 0.9)$ and $d \in [1, 2.2]$ according proposition (2.6), we know that the system has only one positive fixed point. After calculation of the positive fixed point of the system, the flip bifurcation emerges from the fixed point $(9/20, 11/10)$ at $d = 1.598$ and $(a, b, c, d, e) = (1, 0.5, 4, 1.598, 0.9) \in C$. It shows the correctness of proposition (2.6).
From Figure 3 we observe that the fixed point \((9/20, 11/10)\) is stable for \(d < 1.598\), that it loses its stability when \(d = 1.598\) and that an invariant circle appears when the parameter \(d\) exceeds 1.598. Figure 4 is local amplification of

4. Conclusions

In this paper we conclude that the discrete-time prey-predator model (3) has complex dynamics and we show that the unique positive fixed point of model (3) can undergo flip bifurcation and Hopf bifurcation. Moreover system (3) displays much interesting dynamical behaviors, including cascade of periodic-doubling, quasi-periodic orbits and the chaotic sets. These results reveal far richer dynamics of the discrete model compared to the continuous model.

References


