On paranorm $BV_\sigma$ I-convergent sequence spaces defined by an Orlicz function

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Abstract

In this article we introduce and study $\sigma_{BV}(M, p)$, $\alpha_{BV}(M, p)$ and $\infty_{BV}(M, p)$ sequence spaces where $p = (p_k)$ is the sequence of strictly positive real numbers with the help of $BV_\sigma$ space [see [23]] and an Orlicz function $M$. We study some topological and algebraic properties and decomposition theorem. Further we prove some inclusion relations related to these new spaces.

Keywords: Bounded variation, Invariant mean, $\sigma$-Bounded variation, Ideal, Filter, Orlicz function, I-convergence, I-null, Solid space, Sequence algebra, paranorm.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively.

We denote

$$\omega = \{x = (x_k): x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

Let $\ell_\infty$, $c$ and $c_0$ denote the Banach spaces of bounded, convergent and null sequences respectively with norm

$$\|x\| = \sup_k |x_k|$$

Let $v$ denote the space of sequences of bounded variation. That is,

$$v = \left\{x = (x_k): \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty = 0 \right\} \quad (1.1)$$

$v$ is a Banach Space normed by

$$\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}| \quad (see [23])$$

Let $\sigma$ be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional $\phi$ on $\ell_\infty$ is said to be an invariant mean or $\sigma$-mean if and only if

(i) $\phi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$.

(ii) $\phi(e) = 1$ where $e = \{1, 1, 1, \ldots\}$,
(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \left\{x = (x_k) : \lim_{m \to \infty} t_{m,k}(x) = L \text{ uniformly in } k, L = \sigma - \lim x \right\} \quad (1.2)$$

where $m \geq 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + \cdots + x_{\sigma^m(k)}}{m+1} \quad \text{and} \quad t_{-1,k} = 0 \quad (1.3)$$

where $\sigma_m(k)$ denote the $m$-th iterate of $\sigma(k)$ at $k$. In case $\sigma$ is the translation mapping, that is, $\sigma(k) = k+1$, $\sigma$-mean is called a Banach limit (see, [2]) and $V_\sigma$, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (1.2) in which $\sigma(n) = n+1$ was given by Lorentz [19, Theorem 1], and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on $c$ (see, [19]) in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c \quad (1.4),$$

Remark 1.1. In view of above discussion we have $c \subset V_\sigma$.

Theorem 1.2. [23, Theorem 1.1] A $\sigma$-mean extends the limit functional on $c$ in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if $\sigma$ has no finite orbits. That is, if and only if for all $k \geq 0, j \geq 1, \sigma^j(k) \neq k$

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x) \quad (1.5)$$

assuming that $t_{-1,k} = 0$

A straightforward forward calculation shows that (see [22])

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^m j(x_{\sigma}(k) - x_{\sigma}^{-1}(k)), & \text{if } (m \geq 1), \\ x_k, & \text{if } (m = 0) \end{cases} \quad (1.6)$$

For any sequence $x, y$ and scalar $\lambda$, we have

$$\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y)$$

and

$$\phi_{m,k}(\lambda x) = \lambda \phi_{m,k}(x).$$

Definition 1.3. A sequence $x \in \ell_\infty$ is of $\sigma$-bounded variation if and only if

(i) $\sum_{m=0}^{\infty} |\phi_{m,k}(x)|$ converges uniformly in $k$.

(ii) $\lim_{m \to \infty} t_{m,k}(x)$, which must exist, should take the same value for all $k$.

Subsequently invariant means have been studied by Ahmad and Mursaleen [23,1,22], J.P. King [14], Raimi [26], Khan and Ebadullah [12,13] and many others. Mursaleen [23] defined the sequence space $BV_\sigma$, the space of all sequence of $\sigma$-bounded variation as

$$BV_\sigma = \{x \in \ell_\infty : \sum_{m} |\phi_{m,k}(x)| < \infty, \text{uniformly in } k\}$$
**Theorem 1.4.** $BV_\sigma$ is a Banach space normed by
\[
\|x\| = \sup_k \sum |\phi_{m,k}(x)| \quad (c.f.[23],[26],[29],[22])
\]

**Definition 1.5.** A function $M : [0,\infty) \to [0,\infty)$ is said to be an Orlicz function if it satisfies the following conditions

(i) $M$ is continuous, convex and non-decreasing

(ii) $M(0) = 0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$

**Remark 1.6.** If the convexity of an Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

**Remark 1.7.** If $M$ is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

An Orlicz function $M$ is said to satisfy $\Delta_2$–Condition for all values of $u$ if there exists a constant $K > 0$ such that $M(Lu) \leq KL M(u)$ for all values of $L > 1$.

Lindenstrauss and Tzafriri[18] used the idea of an Orlicz function to construct the sequence space
\[
\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0\}.
\] (1.7)

The space $\ell_M$ becomes a Banach space with the norm
\[
\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}
\] (1.8)
which is called an Orlicz sequence space. The space $\ell_M$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $M(t) = t^p$ for $1 < p < \infty$.

Later on some Orlicz sequence spaces were investigated by Parashar and Choudhury [25], Maddox [20], Khan [10], Kamthan and Gupta [9], Bhardwaj and Singh [3], and many others.

**Definition 1.8.** Let $X$ be a linear space. A function $g : X \to \mathbb{R}$ is called paranorm, if for all $x, y \in X$,

(P1) $g(x) = 0$ if $x = \theta$,

(P2) $g(-x) = g(x)$,

(P3) $g(x+y) \leq g(x) + g(y)$,

(P4) If $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ ($n \to \infty$) and $x_n, a \in X$ with $x_n \to a$ ($n \to \infty$) in the sense that $g(x_n - a) \to 0$ ($n \to \infty$), then $g(\lambda_n x_n - \lambda a) \to 0$ ($n \to \infty$).

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value (see,[21]). The notion of paranormed sequence space was studied at the initial stage by Nakano[24]. Later on, it was further investigated by Maddox[20,21], Lascarides[17], Tripathy[30] and many others. A paranorm $g$ for which $g(x) = 0$ implies $x = \theta$ is called a totally paranorm on $X$, and the pair $(X, g)$ is called a totally paranormed space.

Initially, as a generalization of statistical convergence[6,7], the notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salát and Wilczyński ([15,16]). Later on, it was studied by Šalát and Tripathy [30], Hazarika [8,32], Khan and Ebadullah [11,12,13], Demirci [4] and many others.

**Here we give some important definitions.**

**Definition 1.9.** A sequence $x=(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\epsilon > 0$, we have
\[
\lim_{k} \frac{1}{k} \left|\{n \in \mathbb{N} : |x_k - L| \geq \epsilon, n \leq k\}\right| = 0
\]
where vertical lines denote the cardinality of the enclosed set.

**Definition 1.10.** Let $N$ be a non-empty set. Then a family of sets $I \subseteq 2^N$ (power set of $N$) is said to be an ideal if
1) $I$ is additive i.e. $\forall A, B \in I \Rightarrow A \cup B \in I$
2) $I$ is hereditary i.e. $\forall A \in I$ and $B \subseteq A \Rightarrow B \in I$.

**Definition 1.11.** A non-empty family of sets $\mathcal{L}(I) \subseteq 2^N$ is said to be filter on $N$ if and only if
1) $\Phi \notin \mathcal{L}(I)$,
2) $\forall A, B \in \mathcal{L}(I)$ we have $A \cap B \in \mathcal{L}(I)$,
3) $\forall A \in \mathcal{L}(I)$ and $A \subseteq B \Rightarrow B \in \mathcal{L}(I)$.

**Definition 1.12.** An Ideal $I \subseteq 2^N$ is called non-trivial if $I \neq 2^N$.

**Definition 1.13.** A non-trivial ideal $I \subseteq 2^N$ is called admissible if $\{\{x\} : x \in N\} \subseteq I$.

**Definition 1.14.** A non-trivial ideal $I$ is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

**Remark 1.15.** For each ideal $I$, there is a filter $\mathcal{L}(I)$ corresponding to $I$, i.e. $\mathcal{L}(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

**Definition 1.16.** A sequence $x = (x_k) \in \omega$ is said to be $I$-convergent to a number $L$ if for every $\epsilon > 0$, the set $\{k \in N : |x_k - L| \geq \epsilon\} \in I$.
In this case, we write $I - \lim x_k = L$.

**Definition 1.17.** A sequence $x = (x_k) \in \omega$ is said to be $I$-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

**Definition 1.18.** A sequence $x = (x_k) \in \omega$ is said to be $I$-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in I$.

**Definition 1.19.** A sequence $x = (x_k) \in \omega$ is said to be $I$-bounded if there exists some $M > 0$ such that $\{k \in N : |x_k| \geq M\} \in I$.

**Definition 1.20.** A sequence space $E$ said to be solid(normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

**Definition 1.21.** A sequence space $E$ said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$, where $\pi$ is a permutation on $\mathbb{N}$

**Definition 1.22.** A sequence space $E$ said to be sequence algebra if $(x_k) * (y_k) = (x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

**Definition 1.23.** A sequence space $E$ said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all $k$.

**Definition 1.24.** Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 \ldots\} \subseteq \mathbb{N}$ and $E$ be a Sequence space. A $K$-step space of $E$ is a sequence space $\lambda^K_E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

**Definition 1.25.** A canonical pre-image of a sequence $(x_{k_n}) \in \lambda^K_E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\lambda^K_E$ is a set of preimages all elements in $\lambda^K_E$ i.e. $y$ is in the canonical preimage of $\lambda^K_E$ iff $y$ is the canonical preimage of some $x \in \lambda^K_E$. 

Definition 1.26. A sequence space \( E \) is said to be monotone if it contains the canonical preimages of its step space.

Definition 1.27. If \( I = I_f \), the class of all finite subsets of \( N \). Then, \( I \) is an admissible ideal in \( N \) and \( I_f \) convergence coincides with the usual convergence.

Definition 1.28. If \( I = I_\delta = \{ A \subseteq N : \delta(A) = 0 \} \). Then, \( I \) is an admissible ideal in \( N \) and we call the \( I_\delta \)-convergence as the logarithmic statistical convergence.

Definition 1.29. If \( I = I_d = \{ A \subseteq N : d(A) = 0 \} \). Then, \( I \) is an admissible ideal in \( N \) and we call the \( I_d \)-convergence as the asymptotic statistical convergence.

Remark 1.30. If \( I_\delta \lim x_n = l \), then \( I_d \lim x_n = l \)

The following lemmas remained an important tool for the establishment of some results of this article.

Lemma(I). Every solid space is monotone

Lemma(II). Let \( K \in \mathcal{L}(I) \) and \( M \subseteq N \). If \( M \notin I \), then \( M \cap K \notin I \).

Lemma(III). If \( I \subseteq 2^N \) and \( M \subseteq N \). If \( M \notin I \), then \( M \cap N \notin I \).

Khan and K.Ebadullah[18] introduced and studied the following sequence space.

For \( m \geq 0 \)

\[ BV_\sigma^I = \left\{ x = (x_k) \in \omega : \{ k \in N : |\phi_{m,k}(x) - L| \geq \epsilon \} \in I, \text{ for some } L \in \mathbb{C} \right\}. \] (2.1)

2. Main results

In this article we introduce the following classes of sequence spaces:

For \( m \geq 0 \)

\[ BV_\sigma^I(M,p) = \left\{ x = (x_k) \in \omega : \left\{ k \in N : M \left( \frac{\left| \phi_{m,k}(x) - L \right|}{\rho} \right)^p \geq \epsilon \right\} \in I, \text{ for some } L \in \mathbb{C}, \rho > 0 \right\}; \] (2.2)

\[ \omega BV_\sigma^I(M,p) = \left\{ x = (x_k) \in \omega : \left\{ k \in N : M \left( \frac{\left| \phi_{m,k}(x) \right|}{\rho} \right)^p \geq \epsilon \right\} \in I, \text{ for some } \rho > 0 \right\}; \] (2.3)

\[ \ell_\infty(M,p) = \left\{ x = (x_k) \in \omega : \sup_k M \left( \frac{\left| \phi_{m,k}(x) \right|}{\rho} \right)^p < \infty, \text{ for some } \rho > 0 \right\}; \] (2.4)

\[ \omega BV_\sigma^I(M,p) = \left\{ x = (x_k) \in \omega : \left\{ k \in N : \exists K > 0, M \left( \frac{\left| \phi_{m,k}(x) \right|}{\rho} \right)^p \geq K \right\} \in I, \text{ for some } \rho > 0 \right\}. \] (2.5)

We also denote

\[ \mathcal{M}^I_{BV_\sigma}(M,p) = BV_\sigma^I(M,p) \cap \ell_\infty(M,p) \]

and

\[ 0\mathcal{M}^I_{BV_\sigma}(M,p) = 0BV_\sigma^I(M,p) \cap \ell_\infty(M,p). \]
Throughout the article, if required, we denote 
\( \phi_{m,k}(x) = x^k \), \( \phi_{m,k}(y) = y^k \) and \( \phi_{m,k}(z) = z^k \) where \( x, y, z \) are \( (x_k), (y_k) \) and \( (z_k) \) respectively.

**Theorem 2.1.** Let \( p = (p_k) \in l_\infty \). For an Orlicz function \( M \), the classes of sequence \( 0BV_\sigma^I(M, p), BV_\sigma^I(M, p), \) 
\( 0M_{BV_\sigma}^I(M, p) \) and \( M_{BV_\sigma}^I(M, p) \) are the linear spaces.

**Proof.** We shall prove the result for the space \( BV_\sigma^I(M, p) \). Rests will follow similarly.

For, let \( x = (x_k), y = (y_k) \in BV_\sigma^I(M, p) \) be any two arbitrary elements and let \( \alpha, \beta \) are scalars. Now, since \( x = (x_k), y = (y_k) \in BV_\sigma^I(M, p) \). \( \Rightarrow \) For \( \epsilon > 0, \exists \) some +ve numbers \( \rho_1 \) and \( \rho_2 \) such that the sets

\[
A_1 = \left\{ k \in \mathbb{N} : M\left( \frac{|x_k - L_1|}{\rho_1} \right)^{p_k} \geq \frac{\epsilon}{2} \right\} \in I, \text{ for some } L_1 \in \mathbb{C} \tag{2.6}
\]

and

\[
A_2 = \left\{ k \in \mathbb{N} : M\left( \frac{|y_k - L_2|}{\rho_1} \right)^{p_k} \geq \frac{\epsilon}{2} \right\} \in I, \text{ for some } L_2 \in \mathbb{C} \tag{2.7}
\]

Let

\[
\rho_3 = \max\{2 | \alpha | \rho_1, 2 | \beta | \rho_2\} \tag{2.8}
\]

Since, \( M \) is non-decreasing and convex, we have,

\[
M\left( \frac{|(\alpha x_k' + \beta y_k') - (\alpha L_1 + \beta L_2)|}{\rho_3} \right)^{p_k} \leq M\left( \frac{|\alpha| |x_k' - L_1|}{\rho_3} \right)^{p_k} + M\left( \frac{|\beta| |y_k' - L_2|}{\rho_3} \right)^{p_k} \leq M\left( \frac{|x_k' - L_1|}{\rho_1} \right)^{p_k} + M\left( \frac{|y_k' - L_2|}{\rho_2} \right)^{p_k} \tag{2.9}
\]

Therefore, from (2.6), (2.7) and (2.9), we have

\[
\left\{ k \in \mathbb{N} : M\left( \frac{|(\alpha x_k' + \beta y_k') - (\alpha L_1 + \beta L_2)|}{\rho_3} \right)^{p_k} \geq \epsilon \right\} \subseteq A_1 \cup A_2 \in I.
\]

implies that

\[
\left\{ k \in \mathbb{N} : M\left( \frac{|(\alpha x_k' + \beta y_k') - (\alpha L_1 + \beta L_2)|}{\rho_3} \right)^{p_k} \geq \epsilon \right\} \subseteq I
\]

Therefore, \( \alpha(x_k) + \beta(y_k) \in BV_\sigma^I(M, p) \)

But \( x = (x_k), y = (y_k) \in BV_\sigma^I(M, p) \) are the arbitrary elements

Therefore, \( \alpha x_k + \beta y_k \in BV_\sigma^I(M) \), for all \( x = (x_k), y = (y_k) \in BV_\sigma^I(M, p) \) and for all scalars \( \alpha, \beta \)

Hence, \( BV_\sigma^I(M, p) \) is linear

**Theorem 2.2.** Let \( p = (p_k) \in l_\infty \). For an Orlicz function \( M \), the spaces \( M_{BV_\sigma}^I(M, p) \) and \( M_{BV_\sigma}^I(M, p) \) are paranormed spaces, paranormed by

\[
g(x) = \inf_{k \geq 1} \left\{ p_k : \sup_k M\left( \frac{\phi_{m,k}(x)}{\rho} \right)^{p_k} \leq 1, \text{ for some } \rho > 0 \right\}
\]

where \( H = \max\{1, \sup_k p_k\} \).

**Proof.** (P1) Clearly \( g(x) = 0 \) if \( x = \theta \).

(P2) It is obvious that \( g(-x) = g(x) \),
(P3) Let \( x = (x_k) \) and \( y = (y_k) \) be two elements in \( \mathcal{M}^r_{BV} (M, p) \). Now for \( \rho_1, \rho_2 > 0 \), we denote
\[
A_1 = \left\{ \rho_1 : \sup_k M \left( \frac{\varphi_{m,k}(x)}{\rho_1} \right)^{p_k} \leq 1 \right\}
\]
and
\[
A_2 = \left\{ \rho_2 : \sup_k M \left( \frac{\varphi_{m,k}(x)}{\rho_2} \right)^{p_k} \leq 1 \right\}
\]
Let us take \( \rho = \rho_1 + \rho_2 \). Then by using the convexity of \( M \), we have
\[
M \left( \frac{\varphi_{m,k}(x+y)}{\rho} \right) \leq \frac{\rho_1}{\rho_1 + \rho_2} M \left( \frac{\varphi_{m,k}(x)}{\rho_1} \right) + \frac{\rho_2}{\rho_1 + \rho_2} M \left( \frac{\varphi_{m,k}(y)}{\rho_2} \right)
\]
which in terms give us
\[
\sup_k M \left( \frac{\varphi_{m,k}(x+y)}{\rho} \right)^{p_k} \leq 1
\]
and
\[
g(x + y) = \inf \left\{ (\rho_1 + \rho_2)^{\frac{p_k}{k}} : \rho_1 \in A_1, \rho_2 \in A_2 \right\}
\]
\[
\leq \inf \left\{ (\rho_1)^{\frac{p_k}{k}} : \rho_1 \in A_1 \right\} + \inf \left\{ (\rho_2)^{\frac{p_k}{k}} : \rho_2 \in A_1 \right\}
\]
\[
= g(x) + g(y).
\]
(P4) Let \( (\lambda_k) \) be a sequence of scalars with \( \lambda_k \to L \) where \( \lambda_k, L \in \mathbb{C} \) and let \((x_k), x \in \mathcal{M}^r_{BV} (M, p) \) be such that \( g(x_k - x) \to 0 \) as \( k \to \infty \). To prove that \( g(\lambda_k x_k - L x) \to 0 \) as \( k \to \infty \).
We put
\[
A_3 = \left\{ \rho_r > 0 : \sup_k M \left( \frac{\varphi_{m,k}(x_k)}{\rho_r} \right)^{p_k} \leq 1 \right\}
\]
and
\[
A_4 = \left\{ \rho_s > 0 : \sup_k M \left( \frac{\varphi_{m,k}(x_k - x)}{\rho_s} \right)^{p_k} \leq 1 \right\}
\]
By convexity and continuity of \( M \), we observe that
\[
M \left( \frac{\varphi_{m,k}(\lambda_k x_k - L x)}{|\lambda_k - L|_{\rho_r} + |L|_{\rho_s}} \right) \leq M \left( \frac{\varphi_{m,k}(\lambda_k x_k - L x)}{|\lambda_k - L|_{\rho_r} + |L|_{\rho_s}} \right) + M \left( \frac{|\lambda_k - L|_{\rho_r} + |L|_{\rho_s}}{\rho_r} \right) M \left( \frac{\varphi_{m,k}(x_k - x)}{\rho_s} \right)
\]
From the above inequality, it follows that
\[
\sup_k M \left( \frac{\varphi_{m,k}(\lambda_k x_k - L x)}{|\lambda_k - L|_{\rho_r} + |L|_{\rho_s}} \right)^{p_k} \leq 1
\]
and consequently, we have
\[
g(\lambda_k x_k - L x) = \inf \left\{ \left( |\lambda_k - L|_{\rho_r} + |L|_{\rho_s} \right)^{\frac{p_k}{k}} : \rho_r \in A_3, \rho_s \in A_4 \right\}
\]
\[
\leq |\lambda_k - L|^{\frac{p_k}{k}} \inf \left\{ (\rho_r)^{\frac{p_k}{k}} : \rho_r \in A_3 \right\} + |L|^{\frac{p_k}{k}} \inf \left\{ (\rho_s)^{\frac{p_k}{k}} : \rho_s \in A_4 \right\}
\]
\[
\leq \max \left\{ 1, |\lambda_k - L|^{\frac{p_k}{k}} \right\} g(x_k) + \max \left\{ 1, |L|^{\frac{p_k}{k}} \right\} g(x_k - x)
\]
\[
(2.14)
\]
Notice that \( g(x_k) \leq g(x) + g(x_k - x) \) for all \( k \in \mathbb{N} \). Hence by our assumption, the right hand side of (2.14) tends
to 0 as \( k \to \infty \) and the result follows.

For \( \mathcal{M}_{BV_\sigma}^I(M,p) \), the result is similar and hence omitted.

**Theorem 2.3** Let \( M_1 \) and \( M_2 \) be two Orlicz functions and satisfying \( \Delta_2 \) – Condition, then

(a) \( \mathcal{X}(M_2,p) \subseteq \mathcal{X}(M_1M_2,p) \)

(b) \( \mathcal{X}(M_1,p) \cap (M_2,p) \subseteq \mathcal{X}(M_1 + M_2,p) \)

where \( \mathcal{X} = _0BV^I_\sigma, BV^I_\sigma, \mathcal{M}_{BV_\sigma}^I, \mathcal{M}_{BV_\sigma}^I \).

**Proof.** (a). Let \( x = (x_k) \in _0BV^I_\sigma(M_2) \) be any arbitrary element. Let \( \epsilon > 0 \) be given \( \Rightarrow \exists \rho > 0 \) such that

\[
\left\{ k \in \mathbb{N} : M_2 \left( \frac{\| \phi_{m,k}(x) \|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I.
\]

i.e.

\[
\left\{ k \in \mathbb{N} : M_2 \left( \frac{\| x_k \|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I,
\]

(2.15).

Let \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( M_1(t) < \epsilon \), for \( 0 \leq t \leq \delta \).

Let us write

\[
y_k = M_2 \left( \frac{\| x_k \|}{\rho} \right)^{p_k}
\]

and consider

\[
\lim_{k} M_1(y_k) = \lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) + \lim_{y_k > \delta, k \in \mathbb{N}} M_1(y_k).
\]

Now, since \( M_1 \) is an Orlicz function, we have

\[
M_1(\lambda x) \leq \lambda M_1(x)
\]

for all \( \lambda \) with \( 0 < \lambda < 1 \).

Therefore,

\[
\lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) \leq M_1(2) \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k)
\]

(2.16)

For \( y_k > \delta \), we have \( y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta} \).

Now, since \( M_1 \) is non-decreasing and convex, it follows that

\[
M_1(y_k) < M_1(1 + \frac{y_k}{\delta}) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1(\frac{2y_k}{\delta})
\]

Again, since \( M_1 \) satisfies \( \Delta_2 \) – Condition, we have

\[
M_1(y_k) < \frac{1}{2} K \left( \frac{y_k}{\delta} \right) M_1(2) + \frac{1}{2} K \left( \frac{y_k}{\delta} \right) M_1(2).
\]

Thus,

\[
M_1(y_k) < K \left( \frac{y_k}{\delta} \right) M_1(2).
\]

Hence,

\[
\lim_{y_k > \delta, k \in \mathbb{N}} M_1(y_k) \leq \max \{ 1, K \delta^{-1} M_1(2) \} \lim_{y_k > \delta, k \in \mathbb{N}} (y_k).
\]

(2.17)

Therefore, from (2.15), (2.16) and (2.17), it follows that

\[
\left\{ k \in \mathbb{N} : M_1M_2 \left( \frac{\| \phi_{m,k}(x) \|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I,
\]

implies that \( x = (x_k) \in _0BV^I_\sigma(M_1M_2,p) \).

Therefore, \( _0BV^I_\sigma(M_2,p) \subseteq _0BV^I_\sigma(M_1M_2,p) \).

Hence, \( \mathcal{X}(M_2,p) \subseteq \mathcal{X}(M_1M_2,p) \) for \( \mathcal{X} = _0BV^I_\sigma \).

For \( \mathcal{X} = BV^I_\sigma, \mathcal{X} = \mathcal{M}_{BV_\sigma}^I, \mathcal{X} = \mathcal{M}_{BV_\sigma}^I \), the inclusions can be established similarly.

(b). Let \( x = (x_k) \in _0BV^I_\sigma(M_1,p) \cap _0BV^I_\sigma(M_2,p) \). Let \( \epsilon > 0 \) be given. Then there exists \( \rho > 0 \) such that the sets

\[
\left\{ k \in \mathbb{N} : M_1 \left( \frac{\| \phi_{m,k}(x) \|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I,
\]
and

\[
\left\{ k \in \mathbb{N} : M_2 \left( \frac{|\phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I,
\]

Therefore, the inclusion

\[
\left\{ k \in \mathbb{N} : (M_1 + M_2) \left( \frac{|\phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \subseteq \left[ \left\{ k \in \mathbb{N} : M_1 \left( \frac{|\phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \right]
\]

\[
\cup \left\{ k \in \mathbb{N} : M_2 \left( \frac{|\phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \right] \right]
\]

implies that

\[
\left\{ k \in \mathbb{N} : (M_1 + M_2) \left( \frac{|\phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I.
\]

showing that \( x = (x_k) \in _0BV^I_\sigma(M_1 + M_2, p) \)

Hence, \( _0BV^I_\sigma(M_1, p) \cap _0BV^I_\sigma(M_2, p) \subseteq _0BV^I_\sigma(M_1 + M_2, p) \)

For \( X = BV^I_\sigma, X = _0M^I_{BV_\sigma} \) and \( X = _1M^I_{BV_\sigma} \) the inclusions are similar.

For \( M_2(x) = x \) and \( M_1(x) = M(x) \), for all \( x \in [0, \infty) \), we have the following corollary.

**Corollary.** \( X \subseteq X(M, p) \) for \( X = _0BV^I_\sigma, BV^I_\sigma, _0M^I_{BV_\sigma} \) and \( _1M^I_{BV_\sigma} \).

**Theorem 2.4.** For any orlicz function \( M \), the spaces \( _0BV^I_\sigma(M, p) \) and \( _0M^I_{BV_\sigma}(M, p) \) are solid and monotone.

**Proof.** Here we consider \( _0BV^I_\sigma(M, p) \). For \( _0M^I_{BV_\sigma}(M, p) \), the proof shall be similar.

For, let \( x = (x_k) \in _0BV^I_\sigma(M, p) \) be any arbitrary element. \( \Rightarrow \) For \( \epsilon > 0 \), \( \exists \rho > 0 \) with

\[
\left\{ k \in \mathbb{N} : M \left( \frac{|\phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I
\]

Let \( (\alpha_k) \) be a sequence of scalars such that

\[
|\alpha_k| \leq 1, \text{ for all } k \in \mathbb{N}.
\]

Now, since \( M \) is an Orlicz function

We have,

\[
M \left( \frac{|\alpha_k \phi_{m,k}(x)|}{\rho} \right)^{p_k} \leq |\alpha_k| \left( \frac{\phi_{m,k}(x)}{\rho} \right)^{p_k} \leq M \left( \frac{\phi_{m,k}(x)}{\rho} \right)^{p_k}.
\]

Therefore,

\[
\left\{ k \in \mathbb{N} : M \left( \frac{|\alpha_k \phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \subseteq \left\{ k \in \mathbb{N} : M \left( \frac{\phi_{m,k}(x)}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I
\]

implies that

\[
\left\{ k \in \mathbb{N} : M \left( \frac{|\alpha_k \phi_{m,k}(x)|}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I
\]

Thus, \( (\alpha_k x_k) \in _0BV^I_\sigma(M, p) \).

Hence \( _0BV^I_\sigma(M, p) \) is solid

Therefore, by lemma(1) \( _0BV^I_\sigma(M) \) is monotone. Hence the result.
Theorem 2.5. The spaces $\mathcal{M}_{BV}^I(M, p)$ and $0\mathcal{M}_{BV}^I(M, p)$ are not separable.

Proof. By a counter example we prove the result for the space $\mathcal{M}_{BV}^I(M, p)$. For $0\mathcal{M}_{BV}^I(M, p)$, the result follows similarly.

Counter Example.
Let $A$ be an infinite subset of increasing natural numbers such that $A \in I$.
Let
\[ p_k = \begin{cases} 1, & \text{if } k \in A, \\ 2, & \text{otherwise}. \end{cases} \]

Let $P_0 = \{(x_k) : x_k = 0 \text{ or } 1, \text{ for } k \in M \text{ and } x_k = 0, \text{ otherwise}\}$.
Since $A$ is infinite, so $P_0$ is uncountable. Consider the class of open balls $B_1 = \{B(z, \frac{1}{2}) : z \in P_0\}$.
Let $C_1$ be an open cover of $\mathcal{M}_{BV}^I(M, p)$ containing $B_1$.
Since $B_1$ is uncountable, so $C_1$ cannot be reduced to a countable subcover for $\mathcal{M}_{BV}^I(M, p)$. Thus $\mathcal{M}_{BV}^I(M, p)$ is not separable.

Theorem 2.6. Let $H = \sup_k p_k < \infty$ and $I$ an admissible ideal. Then the following are equivalent.

(a) $x = (x_k) \in BV^I_\sigma(M, p)$;
(b) there exists $y = (y_k) \in BV_\sigma(M, p)$ such that $x_k = y_k$, for a.a.k.r.I;
(c) there exists $y = (y_k) \in BV_\sigma(M, p)$ and $z = (z_k) \in 0BV^I_\sigma(M, p)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and
\[ \{k \in \mathbb{N} : M\left(\frac{|y_k - L|}{\rho}\right)^{p_k} \geq \epsilon\} \in I; \]
(d) there exists a subset $K = \{k_1 < k_2, \ldots\}$ of $\mathbb{N}$ such that $K \in \mathcal{L}(I)$ and
\[ \lim_{n \to \infty} M\left(\frac{|z_{k_n} - L|}{\rho}\right)^{p_{k_n}} = 0. \]

Proof. (a) implies (b). Let $x = (x_k) \in BV^I_\sigma(M, p)$. Then there exists $L \in \mathbb{C}$ such that
\[ \{k \in \mathbb{N} : M\left(\frac{|x_k - L|}{\rho}\right)^{p_k} \geq \epsilon\} \in I. \]

Let $(m_t)$ be an increasing sequence with $m_t \in \mathbb{N}$ such that
\[ \{k \leq m_t : M\left(\frac{|x_k - L|}{\rho}\right)^{p_k} \geq t^{-1}\} \in I. \]

Define a sequence $(y_k)$ as
\[ y_k = x_k, \text{ for all } k \leq m_1. \]

For $m_t < k \leq m_{t+1}$, $t \in \mathbb{N}.$
\[ y_k = \begin{cases} x_k, & \text{if } M\left(\frac{|x_k - L|}{\rho}\right)^{p_k} < t^{-1} \\ L, & \text{otherwise}. \end{cases} \]

Then $y = (y_k) \in BV_\sigma(M, p)$ and form the following inclusion
\[ \{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : M\left(\frac{|x_k - L|}{\rho}\right)^{p_k} \geq \epsilon\} \in I. \]

We get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c). For $(x_k) \in BV^I_\sigma(M, p)$. Then there exists $(y_k) \in BV_\sigma(M, p)$ such that $x_k = y_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $K \in I$.
Define a sequence $(z_k)$ as
\[ z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise}. \end{cases} \]
Then $z_k \in _0BV^I_\sigma(M, p)$ and $y_k \in BV_\sigma(M, p)$.

(c) implies (d). Suppose (c) holds. Let $\epsilon > 0$ be given. Let $P_1 = \{ k \in \mathbb{N} : M\left(\left|\frac{x_k' - L}{\rho}\right|^{p_k}\right) \geq \epsilon \} \in I$ and

$$K = P_1^c = \{ k_1 < k_2 < k_3 < \ldots \} \in L(I).$$

Then, we have $\lim_{n \to \infty} M\left(\left|\frac{x_k' - L}{\rho}\right|^{p_k}\right)_{k \leq n} = 0$.

(d) implies (a). Let $K = \{ k_1 < k_2 < k_3 < \ldots \} \in L(I)$ and $\lim_{n \to \infty} M\left(\left|\frac{x_k' - L}{\rho}\right|^{p_k}\right) = 0$.

Then, for any $\epsilon > 0$, and Lemma (II), we have

$$\left\{ k \in \mathbb{N} : M\left(\left|\frac{x_k' - L}{\rho}\right|^{p_k}\right) \geq \epsilon \right\} \subseteq K^c \cup \left\{ k \in \mathbb{N} : M\left(\left|\frac{x_k' - L}{\rho}\right|^{p_k}\right) \geq \epsilon \right\} \in I$$

implies that

$$\left\{ k \in \mathbb{N} : M\left(\left|\frac{x_k' - L}{\rho}\right|^{p_k}\right) \geq \epsilon \right\} \in I$$

Therefore, $(x_k) \in BV^I_\sigma(M, p)$.

Hence the result.

**Theorem 2.7.** Let $h = \inf_k p_k$ and $H = \sup_k p_k$. Then, the following results are equivalent. (a) $H < \infty$ and $h > 0$.

(b) $0BV^I_\sigma(M, p) = BV^I_\sigma$.

**Proof.** Suppose that $H < \infty$ and $h > 0$, then the inequalities $\min\{1, s^k\} \leq s^{p_k} \leq \max\{1, s^H\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$.

Therefore the equivalent of (a) and (b) is obvious.

**Theorem 2.8.** Let $p = (q_k)$ and $q = (q_k)$ be two sequences of positive real numbers. Then $0M^I_{BV_\sigma}(M, p) \supseteq 0M^I_{BV_\sigma}(M, q)$ if and only if $\lim \inf_{k \to K} \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

**Proof.** Let $\lim \inf_{k \to K} \frac{p_k}{q_k} > 0$. and $(x_k) \in 0M^I_{BV_\sigma}(M, p)$. Then, there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$.

Since $(x_k) \in 0M^I_{BV_\sigma}(M, p)$.

For a given $\epsilon > 0$, $\exists \rho > 0$ such that

$$B_0 = \left\{ k \in \mathbb{N} : M\left(\left|\frac{x_k'}{\rho}\right|^{p_k}\right) \geq \epsilon \right\} \in I.$$ 

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$.

Then, for all sufficiently large $k \in G_0$,

$$\left\{ k \in \mathbb{N} : M\left(\left|\frac{x_k'}{\rho}\right|^{p_k}\right) \geq \epsilon \right\} \subseteq \left\{ k \in \mathbb{N} : M\left(\left|\frac{x_k'}{\rho}\right|^{\beta q_k}\right) \geq \epsilon \right\} \in I.$$ 

implies that

$$\left\{ k \in \mathbb{N} : M\left(\left|\frac{x_k'}{\rho}\right|^{p_k}\right) \geq \epsilon \right\} \in I$$

Therefore $(x_k) \in 0M^I_{BV_\sigma}(M, p)$.

Converse part of the result follows obviously.
Theorem 2.9. Let \( p = (p_k) \) and \( q = (q_k) \) be two sequences of positive real numbers. Then
\[
0\mathcal{M}^I_{BV_c}(M, q) \supseteq 0\mathcal{M}^I_{BV_c}(M, p)
\]
if and only if \( \liminf_{k \to k} \frac{q_k}{p_k} > 0 \), where \( K^c \subseteq \mathbb{N} \) such that \( K \in I \).

Proof. The proof follows similarly as the proof of Theorem 2.8.

Theorem 2.10. Let \( p = (p_k) \) and \( q = (q_k) \) be two sequences of positive real numbers. Then \( 0\mathcal{M}^I_{BV_c}(M, p) = 0\mathcal{M}^I_{BV_c}(M, q) \) if and only if \( \liminf_{k \in K} \frac{p_k}{q_k} > 0 \), and \( \liminf_{k \in K} \frac{q_k}{p_k} > 0 \), where \( K^c \subseteq \mathbb{N} \) such that \( K \in I \).

Proof. On combining Theorem 2.9 and 2.10 we get the required result.

Theorem 2.11. The set \( \mathcal{M}^I_{BV_c}(M, p) \) is closed subspace of \( \ell_{\infty}(M, p) \).

Proof. Let \( (x^{(i)}_k) \) be a Cauchy sequence in \( \mathcal{M}^I_{BV_c}(M, p) \) such that \( x^{(i)} \to x \).

We show that \( x \in \mathcal{M}^I_{BV_c}(M, p) \)

Since \( (x^{(i)}_k) \in \mathcal{M}^I_{BV_c}(M, p) \), then there exists a sequence \( a_i \) and \( \rho > 0 \) such that
\[
\{k \in \mathbb{N} : M\left( \left| \frac{(x^{(i)}_k)^j - a_i}{\rho} \right| \right)^{p_k} \geq \epsilon \} \subseteq I
\]

We need to show that
1. \( (a_i) \) converges to \( a \).
2. If \( U = \{k \in \mathbb{N} : M\left( \left| (x^{(i)}_k)^j - a_i \right| / \rho \right)^{p_k} < \epsilon \} \), then \( U^c \in I \).

1. Since \( (x^{(i)}_k) \) is Cauchy sequence in \( \mathcal{M}^I_{BV_c}(M, p) \) \( \Rightarrow \) for a given \( \epsilon > 0 \), there exists \( k_0 \in \mathbb{N} \) such that
\[
\sup_k M\left( \left| \frac{(x^{(i)}_k)^j - (x^{(j)}_k)^j}{\rho} \right| \right)^{p_k} < \frac{\epsilon}{3}, \text{ for all } i, j \geq k_0.
\]

For \( \epsilon > 0 \), we have
\[
B_{ij} = \left\{ k \in \mathbb{N} : M\left( \left| \frac{(x^{(i)}_k)^j - (x^{(j)}_k)^j}{\rho} \right| \right)^{p_k} < \frac{\epsilon}{3} \right\}
\]
\[
B_i = \left\{ k \in \mathbb{N} : M\left( \left| \frac{(x^{(i)}_k)^j - a_i}{\rho} \right| \right)^{p_k} < \frac{\epsilon}{3} \right\}
\]
\[
B_j = \left\{ k \in \mathbb{N} : M\left( \left| \frac{(x^{(j)}_k)^j - a_j}{\rho} \right| \right)^{p_k} < \frac{\epsilon}{3} \right\}
\]

Then, \( B_{ij}^c, B_i^c, B_j^c \in I \)

Let \( B^c = B_{ij}^c \cup B_i^c \cup B_j^c \), where \( B = \left\{ k \in \mathbb{N} : M\left( \left| \frac{a_i - a_j}{\rho} \right| \right)^{p_k} < \epsilon \right\} \).

Then, \( B^c \in I \).

We choose \( k_0 \in B^c \).

Then for each \( i, j \geq k_0 \), we have
\[
\left\{ k \in \mathbb{N} : M\left( \left| \frac{a_i - a_j}{\rho} \right| \right)^{p_k} < \epsilon \right\} \supseteq \left\{ k \in \mathbb{N} : M\left( \left| \frac{a_i - a_i}{\rho} \right| \right)^{p_k} < \frac{\epsilon}{3} \right\}
\]
\[
\cap \left\{ k \in \mathbb{N} : M\left( \left| \frac{(x^{(i)}_k)^j - a_i}{\rho} \right| \right)^{p_k} < \frac{\epsilon}{3} \right\}
\]
Therefore, for each 
\( k \in \mathbb{N} : M \left( \left| \frac{a_j - (x_j(k))^j}{\rho} \right|^{p_k} \right) < \frac{\epsilon}{3} \)
implies that 
\((a_i)\) is a Cauchy sequence of scalars in \( C \), so there exists a scalar \( a \) in \( C \) such that \( a_i \to a \), as \( n \to \infty \).

(2) Let \( 0 < \delta < 1 \) be given. Then we show that if 
\[ U = \{ k \in \mathbb{N} : M \left( \left| \frac{(x(k)^j)^j - a}{\rho} \right|^{p_k} \right) \leq \epsilon \} \], then \( U^c \in I \).
Since \( x(i) \to x \), then there exists \( q_0 \in \mathbb{N} \) such that 
\[ P = \left\{ k \in \mathbb{N} : M \left( \left| \frac{(x_k(q_0))^j - x_k}{\rho} \right|^{p_k} \right) < \left( \frac{\delta}{3D} \right)^H \right\} \] (2.21)
where \( D = \max\{1, 2^{G-1}\} \), \( G = \sup p_k \geq 0 \) and \( H = \max\{1, \sup p_k \} \)
implies \( P^c \in I \).
The number \( q_0 \) can be chosen that together with (2.21), we have 
\[ Q = \left\{ k \in \mathbb{N} : M \left( \left| \frac{a_{q_0} - a}{\rho} \right|^{p_k} \right) < \left( \frac{\delta}{3D} \right)^H \right\} \]
such that \( Q^c \in I \).
Since 
\[ \left\{ k \in \mathbb{N} : M \left( \left| \frac{(x_k(q_0)^j)^j - a_{q_0}}{\rho} \right|^{p_k} \right) \geq \delta \right\} \in I. \]
Then, we have a subset \( S \) of \( \mathbb{N} \) such that \( S^c \in I \), where
\[ S = \left\{ k \in \mathbb{N} : M \left( \left| \frac{(x_k(q_0)^j)^j - a_{q_0}}{\rho} \right|^{p_k} \right) < \left( \frac{\delta}{3D} \right)^H \right\}. \]
Let \( U^c = P^c \cup Q^c \cup S^c \), where 
\[ U = \left\{ k \in \mathbb{N} : M \left( \left| \frac{x_k - a}{\rho} \right|^{p_k} \right) < \delta \right\} \]
Therefore, for each \( k \in U^c \), we have 
\[ \left\{ k \in \mathbb{N} : M \left( \left| \frac{(x_k(q_0)^j)^j - a_{q_0}}{\rho} \right|^{p_k} \right) < \delta \right\} \subseteq \left[ \left\{ k \in \mathbb{N} : M \left( \left| \frac{(x_k(q_0)^j - x_k)^j}{\rho} \right|^{p_k} \right) < \left( \frac{\delta}{3D} \right)^H \right\} \right. \]
\[ \cap \left\{ k \in \mathbb{N} : M \left( \left| \frac{a_{q_0} - a}{\rho} \right|^{p_k} \right) < \left( \frac{\delta}{3D} \right)^H \right\} \]
\[ \cap \left\{ k \in \mathbb{N} : M \left( \left| \frac{(x_k(q_0)^j)^j - a_{q_0}}{\rho} \right|^{p_k} \right) < \left( \frac{\delta}{3D} \right)^H \right\} \].
Then the result follows.
Since the inclusions \( \mathcal{M}_{BV}^I(M,p) \subset \ell_\infty(M,p) \) and \( \mathcal{M}_{BV}^\ell(M,p) \subset \ell_\infty(M,p) \) are strict so in view of Theorem (2.11) we have the following result.

**Theorem 2.12.** The spaces \( \mathcal{M}_{BV}^I(M,p) \) and \( \mathcal{M}_{BV}^\ell(M,p) \) are nowhere dense subsets of \( \ell_\infty(M,p) \).

**Theorem 2.13.** For an Orlicz function \( M \), the spaces \( \mathcal{M}_{BV}^I(M,p) \) and \( \mathcal{M}_{BV}^\ell(M,p) \) are sequence algebra.

**Proof.** Here we consider \( \mathcal{M}_{BV}^I(M,p) \). For the other result the proof is similar.
Let \( x = (x_k), y = (y_k) \in \mathcal{M}_{BV}^I(M,p) \) be any two arbitrary elements.
⇒ ∃ ρ₁, ρ₂ > 0 such that
\[
\left\{ k \in \mathbb{N} : M \left( \frac{|φ_{m,k}(x)|}{ρ_1} \geq ϵ \right)^{p_k} \right\} \in I. \tag{2.22}
\]
and
\[
\left\{ k \in \mathbb{N} : M \left( \frac{|φ_{m,k}(y)|}{ρ_1} \geq ϵ \right)^{p_k} \right\} \in I. \tag{2.23}
\]

Let ρ = ρ₁ρ₂ > 0
Then, it is obvious from (2.22) and (2.23) that
\[
\left\{ k \in \mathbb{N} : M \left( \frac{|φ_{m,k}(x)φ_{m,k}(y)|}{ρ} \geq ϵ \right)^{p_k} \right\} \in I.
\]
which further implies that \((x_k, y_k) = (x_k y_k) \in _0BV^I_σ(M, p)\)
Hence, \(_0BV^I_σ(M, p)\) is a Sequence algebra.

**Theorem 2.11.** Let \(M\) be an Orlicz function. Then, \(_0BV^I_σ(M, p) \subset BV^I_σ(M, p) \subset ∞BV^I_σ(M, p)\).

**Proof.** Let \(M\) be an Orlicz function. Then, we have to show that \(_0BV^I_σ(M, p) \subset BV^I_σ(M, p) \subset ∞BV^I_σ(M, p)\).
Firstly, \(_0BV^I_σ(M) \subset BV^I_σ(M)\) is obvious.
Let \(x = (x_k) \in BV^I_σ(M, p)\). Then there exists \(L \in \mathbb{C}\) and \(ρ > 0\) such that
\[
\left\{ k \in \mathbb{N} : M \left( \frac{|x_k - L|}{ρ} \right)^{p_k} \geq ϵ \right\} \in I.
\]
That is
\[
I - \lim M \left( \frac{|x_k - L|}{ρ} \right)^{p_k} = 0.
\]
Therefore, we have
\[
M \left( \frac{|x_k|}{2ρ} \right)^{p_k} \leq \frac{1}{2} M \left( \frac{|x_k - L|}{ρ} \right)^{p_k} + \frac{1}{2} M \left( \frac{|L|}{ρ} \right)^{p_k}.
\]
Taking supremum over \(k\) both sides, we get \(x = (x_k) \in ∞BV^I_σ(M, p)\).
Hence, \(_0BV^I_σ(M, p) \subset BV^I_σ(M, p) \subset ∞BV^I_σ(M, p)\).

**Theorem 2.15.** If \(I\) is not maximal and \(I \neq I_f\). Then, the space \(_0BV^I_σ(M, p)\) and \(BV^I_σ(M, p)\) are not symmetric.

**Proof.** Let \(A ∈ I\) be any infinite set and \(M(x) = x\), for all \(x \in [0, ∞)\).
Define a sequence \((x_k)\) as
\[
x_k = \begin{cases} 
1, & \text{if } k \in A, \\
0, & \text{otherwise}.
\end{cases}
\]

Then, it is clear that \((x_k) \in _0BV^I_σ(M, p) \subset BV^I_σ(M, p)\)
Let \(K ⊆ \mathbb{N}\) be such that \(K \notin I\) and \(\mathbb{N} \setminus K \notin I\).
Let \(φ : K \rightarrow A\) and \(ψ : K^c \rightarrow A^c\) be bijective maps. Then, the mapping \(π : \mathbb{N} \rightarrow \mathbb{N}\) defined by
\[
π(k) = \begin{cases} 
φ(k), & \text{if } k ∈ K, \\
ψk, & \text{otherwise}.
\end{cases}
\]
is a permutation on \(\mathbb{N}\)
But \((x_\pi(k)) \notin BV^I_\sigma(M,p)\) and hence \((x_\pi(k)) \notin 0BV^I_\sigma(M,p)\) showing that
\[BV^I_\sigma(M,p) \text{ and } 0BV^I_\sigma(M,p)\]
are not symmetric sequence spaces.

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**References**


