A best proximity point theorem for generalized Mizoguchi- Takahashi contractions

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Abstract

The purpose of this paper is to provide sufficient conditions for the existence of a unique best proximity point for generalized Mizoguchi- Takahashi contractions. Our paper provides an extension of a result due to Gordji and Ramezani[3].

Keywords: Fixed point, best proximity point, P-property, Mizoguchi- Takahashi contractions.

1. Introduction

Let \((X, d)\) be a metric space. Denote by \(P(X)\) the set of all nonempty subsets of \(X\) and \(CB(X)\) the family of all nonempty closed and bounded subsets of \(X\). A point \(x\) in \(X\) is a fixed point of a multivalued map \(T : X \rightarrow P(X)\), if \(x \in Tx\). Nadler [5] extended the Banach contraction principle to multivalued mappings.

Theorem 1.1 (5) Let \((X, d)\) be a complete metric spaces and let \(T : X \rightarrow CB(X)\) be a multivalued map. Assume that there exists \(r \in [0, 1)\) such that

\[ H(Tx, Ty) \leq rd(x, y) \]

for all \(x, y \in X\), where \(H\) is the Haudorff metric with respect to \(d\). Then \(T\) has a fixed point.

The fixed point theory for multivalued mappings developed rapidly after the publication of Nadler’s paper [5] in which he established a multivalued version of Banach’s contraction principle. A huge number of generalizations of this principle appear in the literature. Particularly, the following generalization of Nadler’s fixed point theorem due to Mizoguchi- Takahashi [4].

Theorem 1.2 (4) Let \((X, d)\) be a complete metric space and \(T : X \rightarrow CB(X)\) be a multivalued map. Assume that

\[ H(Tx, Ty) \leq \phi(d(x, y))d(x, y) \]  \hspace{1cm} (1)

for all \(x, y \in X\), where \(\phi\) is a function from \([0, \infty)\) into \([0, 1)\) satisfying \(\limsup_{s \to +t} \phi(s) < 1\) for all \(t \geq 0\). Then \(T\) has a fixed point.
Recently, Amini-Harandi and O'Regan [1] obtained a nice generalization of Mizoguchi and Takahashi’s fixed point theorem. Throughout the article, let \( \Psi \) be the family of all functions \( \psi : [0, \infty) \to [0, \infty) \) satisfying the following conditions:
(a) \( \psi(s) = 0 \iff s = 0 \),
(b) \( \psi \) is nondecreasing,

We denote by \( \Phi \) the set of all functions \( \phi : [0, \infty) \to [0,1) \) satisfying \( \limsup_{t \to s^+} \phi(r) < 1 \) for all \( t \geq 0 \).

Amini-Harandi and O’Regan generalized the Mizoguchi-Takahashi contraction condition (1) as follows:

**Theorem 1.3 (1)** Let \( (X, d) \) be a complete metric space and \( T : X \to CB(X) \) be a multivalued map. Assume that

\[
\phi(H(Tx, Ty)) \leq \phi(\psi(d(x, y)))\psi(d(x, y))
\]

for all \( x, y \in X \), where \( \psi \in \Psi \) is lower semicontinuous with \( \limsup_{s \to 0^+} \frac{\phi(s)}{\psi(s)} < \infty \) and \( \phi \in \Phi \). Then \( T \) has a fixed point.

Very recently, Gordji and Ramezani [3] established a new fixed point theorem for a self map \( T : X \to X \) satisfying a generalized Mizoguchi-Takahashi’s condition in the setting of ordered metric spaces. The main result in [3] is the following.

**Theorem 1.4 (3)** Let \( (X, d, \preceq) \) be a complete ordered metric space and \( T : X \to X \) an increasing mapping such that there exists an element \( x_0 \in X \) with \( x_0 \preceq Tx_0 \). Suppose that there exists a lower semicontinuous function \( \psi \in \Psi \) and \( \phi \in \Phi \) such that

\[
\psi(d(Tx, Ty)) \leq \phi(\psi(d(x, y)))\psi(d(x, y)).
\]

for all \( x, y \in X \) such that \( x \) and \( y \) are comparable. Assume that either \( T \) is continuous or \( X \) is such that the following holds: any \( \preceq \)-nondecreasing sequence \( \{x_n\} \) with \( x_n \to x \) implies \( x_n \preceq x \) for all \( n \). Then \( T \) has a fixed point.

The aim of this paper is to give a generalization of the Theorem 1.4 by considering a non-self map \( T \).

## 2. Preliminary notes

First, we present a brief discussion about a best proximity point.

Let \( A \) be a nonempty subset of a metric space \( (X, d) \) and \( T : A \to X \) be a mapping. The solutions of the equation \( Tx = x \) are fixed point of \( T \). Consequently, \( T(A) \cap A \neq \emptyset \) is a necessary condition for the existence of a fixed point for the operator \( T \). If this necessary condition does not hold, then \( d(x, Tx) > 0 \) for any \( x \in A \) and the mapping \( T : A \to X \) does not have any fixed point. In this setting, our aim is to find an element \( x \in A \) such that \( d(x, Tx) \) is minimum in some sense. A point \( x \) in \( A \) for which \( d(x, Tx) = d(A, B) \) is called a best proximity point of \( T \).

In our context, we consider two nonempty subsets \( A \) and \( B \) of a complete metric space and a mapping \( T : A \to B \) satisfying a generalized Mizoguchi-Takahashi’s condition and find a best proximity point of \( T \). We give an example to support our result.

Let \( A \) and \( B \) be two nonempty subsets of a metric space \( (X, d) \). We denote by \( A_0 \) and \( B_0 \) the following sets:

\[
A_0 = \{ x \in A : d(x, y) = d(A, B) \ for \ some \ y \in B \},
\]

\[
B_0 = \{ y \in B : d(x, y) = d(A, B) \ for \ some \ x \in A \},
\]

where \( d(A, B) = \inf \{d(x, y) : x \in A \ and \ y \in B\} \).

In [6] authors present sufficient conditions which determine when the sets \( A_0 \) and \( B_0 \) are nonempty.

**Definition 2.1** Let \( A, B \) be two nonempty subsets of a metric space \( (X, d) \). A mapping \( T : A \to B \) is said to be a generalized Mizoguchi-Takahashi contractions if there exist \( \phi \in \Phi \) and \( \psi \in \Psi \) such that

\[
\psi(d(Tx, Ty)) \leq \phi(\psi(d(x, y)))\psi(d(x, y))
\]

for any \( x, y \in A \).
Definition 2.2 (6) Let \((A, B)\) be a pair of nonempty subsets of a metric space \((X, d)\) with \(A_0 \neq \emptyset\). Then the pair \((A, B)\) is said to have the P-property if and only if for any \(x_1, x_2 \in A_0\) and \(y_1, y_2 \in B_0\),

\[
\begin{cases}
    d(x_1, y_1) = d(A, B) \\
    d(x_2, y_2) = d(A, B)
\end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2).
\]

3. Main results

Theorem 3.1 Let \((A, B)\) be a pair of nonempty closed subsets of a complete metric space \((X, d)\) such that \(A_0\) is nonempty. Let \(T : A \to B\) be a continuous generalized Mizoguchi- Takahashi contraction mapping satisfying \(T(A_0) \subset B_0\). Suppose that the pair \((A, B)\) has the P-property. Then there exists a unique \(x^*\) in \(A\) such that \(d(x^*, Tx^*) = d(A, B)\).

Proof. Since \(A_0\) is nonempty, we take \(x_0 \in A\). As \(Tx_0 \in T(A_0) \subset B_0\), we can find \(x_1 \in A_0\) such that \(d(x_1, Tx_0) = d(A, B)\). Similarly, since \(Tx_1 \in T(A_0) \subset B_0\), there exists \(x_2 \in A_0\) such that \(d(x_2, Tx_1) = d(A, B)\). Repeating this process, we can get a sequence \(\{x_n\}\) in \(A_0\) satisfying

\[d(x_{n+1}, Tx_n) = d(A, B) \text{ for any } n \in N\]

Since \((A, B)\) has the P-property, we have that

\[d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \text{ for any } n \in N.\]

Taking into account that \(T\) is a generalized Mizoguchi- Takahashi contraction, for any \(n \in N\), we have that

\[
\psi(d(x_n, x_{n+1})) \leq \phi(\psi(d(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n))
\leq \psi(d(x_{n-1}, x_n))
\]

Since \(\psi\) is nondecreasing, we obtain

\[d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)\]

This means \(d(x_n, x_{n+1})\) is a non-increasing sequence of positive real numbers. Hence there exists \(\mu \geq 0\) such that

\[\lim_{n \to \infty} d(x_n, x_{n+1}) = \mu\]

Since \(\phi \in \Phi\), we have \(\limsup_{r \to \mu} \phi(r) < 1\). Then, there exist \(\alpha \in [0, 1)\) and \(\epsilon > 0\) such that \(\phi(r) \leq \alpha\) for all \(r \in [\mu, \mu + \epsilon)\). We can take \(n_0 \in N\) such that \(\mu \leq d(x_n, x_{n+1}) \leq \mu + \epsilon\) for all \(n \geq n_0\). Then for all \(n \geq n_0\), we have

\[
\psi(d(x_n, x_{n+1})) \leq \phi(\psi(d(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n))
\]

Letting \(r \to \infty\) in the above inequality, we obtain that

\[\psi(\mu) \leq \alpha \psi(\mu)\]  \(\text{(2)}\)

Since \(\alpha \in [0, 1)\), this implies that \(\mu = 0\). Thus, we have

\[\lim_{n \to \infty} d(x_n, x_{n+1}) = 0\]  \(\text{(3)}\)

Now we claim that the sequence \(\{x_n\}\) is a Cauchy sequence. Since \(\lim_{n \to \infty} d(x_n, x_{n+1}) = 0\), it is sufficient to prove that \(\{x_{2n}\}\) is Cauchy sequence.

Suppose on the contrary that \(\{x_{2n}\}\) is not a Cauchy sequence. Then there exist \(\epsilon > 0\) and subsequences \(\{x_{2n_k}\}\) and \(\{x_{2m_k}\}\) of \(\{x_{2n}\}\) such that \(n_k > m_k > k\) and

\[d(x_{2m_k}, x_{2n_k}) \geq \epsilon\]  \(\text{(4)}\)
and
\[ d(x_{2m_k}, x_{2n_{k-2}}) < \epsilon \]

Now, from (4) and the triangle inequality, we get
\[
\epsilon \leq d(x_{2m_k}, x_{2n_k}) \\
\leq d(x_{2m_k}, x_{2n_{k-2}}) + d(x_{2n_{k-2}}, x_{2n_{k-1}}) + d(x_{2n_{k-1}}, x_{2n_k})
\]

Letting \( k \to \infty \) and using (3), we get
\[
\lim_{k \to \infty} d(x_{2m_k}, x_{2n_k}) = \epsilon
\]

By the fact
\[
|d(x_{2m_k}, x_{2n_{k+1}}) - d(x_{2m_k}, x_{2n_k})| \leq d(x_{2n_k}, x_{2n_{k+1}})
\]

using (3) and (6), we obtain
\[
\lim_{k \to \infty} d(x_{2m_k}, x_{2n_{k+1}}) = \lim_{k \to \infty} d(x_{2m_k}, x_{2n_{k+1}}) = \epsilon
\]

Moreover, from
\[
|d(x_{2m_{k-1}}, x_{2n_{k+1}}) - d(x_{2m_{k-1}}, x_{2n_k})| \leq d(x_{2n_k}, x_{2n_{k+1}})
\]

and combining with (3) and (9), we conclude that
\[
\lim_{k \to \infty} d(x_{2m_{k-1}}, x_{2n_{k+1}}) = \epsilon
\]

from (9), we conclude that
\[
\lim_{k \to \infty} d(Tx_{2m_{k-1}}, Tx_{2n_k}) = \epsilon
\]

Letting \( k \to \infty \) and using (2) and (12), we have
\[
\psi(\epsilon) \leq \alpha \psi(\epsilon)
\]
a contradiction. Therefore, \( \{x_n\} \) is a Cauchy sequence. Since \( \{x_n\} \subset A \) and A is closed subset of a complete metric space \((X,d)\), we can find \( x^* \in A \) such that \( x_n \to x^* \).

Since T is continuous, we have \( Tx_n \to Tx^* \). Taking into account that the sequence \( d(x_{n+1}, Tx_n) \) is a constant sequence with value \( d(A,B) \), we deduce
\[ d(x^*, Tx^*) = d(A,B). \]

This means that \( x^* \) is a best proximity point of T. For uniqueness, suppose that \( x_1 \) and \( x_2 \) are two best proximiy points of T with \( x_1 \neq x_2 \). This means that
\[
d(x_1, Tx_1) = d(A,B) \\
d(x_2, Tx_2) = d(A,B)
\]

Using the P-property, we have
\[
d(x_1, x_2) = d(Tx_1, Tx_2)
\]

Again, T is generalized Mizoguchi- Takahashi contraction, we have
\[
\psi(d(x_1, x_2)) = \psi(Tx_1, Tx_2) \leq \phi(\psi(d(x_1, x_2))) \psi(d(x_1, x_2)) \leq \alpha \psi(d(x_1, x_2))
\]
a contradiction. Therefore, \( x_1 = x_2 \).
Corollary 3.2 Let \((A, B)\) be a pair of nonempty closed subsets of a complete metric space \((X, d)\) such that \(A_0\) is nonempty. Let \(T : A \rightarrow B\) be a continuous mapping satisfying \(T(A_0) \subset B_0\), and Mizoguchi- Takahashi contraction condition
\[
d(Tx, Ty) \leq \phi(d(x, y))d(x, y)
\]
for any \(x, y \in A\). Suppose that the pair \((A, B)\) has the P-property. Then there exists a unique \(x^*\) in \(A\) such that \(d(x^*, Tx^*) = d(A, B)\).

Example 3.3 Consider \(X = \mathbb{R}^2\) with the usual metric. Let \(A\) and \(B\) be the subsets of \(X\) defined by
\[
A = \{0\} \times [0, \infty) \text{ and } B = \{1\} \times [0, 1).
\]
Obviously \(d(A, B) = 1\) and \(B\) is not closed subset of \(X\).

Note that \(A_0 = 0 \times [0, 1)\) and \(B_0 = B\).

We consider the mapping \(T : A \rightarrow B\) defined as
\[
d(0, x) = \left(1, \frac{x}{1 + x}\right) \text{ for any } (0, x) \in A.
\]

In the sequel, we check that \(T\) is generalized Mizoguchi- Takahashi contraction. In fact, for \((0, x), (0, y) \in A\) with \(x \neq y\), we have
\[
d(T(0, x), T(0, y)) = d((1, \frac{x}{1 + x}), (1, \frac{y}{1 + y}))
\]
\[
= \left| \frac{x}{1 + x} - \frac{y}{1 + y} \right|
\]
\[
= \left| \frac{x - y}{(1 + x)(1 + y)} \right|
\]
\[
\leq \left| \frac{x - y}{1 + |x - y|} \right|
\]
\[
= \frac{\psi(d((0, x), (0, y)))}{d((0, x), (0, y))}.
\]

Where \(\psi(t) = t\) for \(t > 0\) and \(\phi(t) = \frac{1}{1 + t}\) with \(\limsup_{t \to t^+} \phi(r) < 1\) for \(t \geq 0\).

Notice that the pair \((A, B)\) satisfies the P-property. Indeed, if
\[
d((0, x_1), (1, y_1)) = \sqrt{1 + (x_1 - y_1)^2} = d(A, B) = 1,
\]
\[
d((0, x_2), (1, y_2)) = \sqrt{1 + (x_2 - y_2)^2} = d(A, B) = 1,
\]
then \(x_1 = y_1\) and \(x_2 = y_2\) and consequently,
\[
d((0, x_1), (0, x_2)) = |x_1 - x_2| = |y_1 - y_2| = d((1, y_1), (1, y_2)).
\]

By Theorem 3.1, \(T\) has a unique best proximity point. Obviously, the point \((0, 0) \in A\) is a unique best proximity point for \(T\), since
\[
d((0, 0), T(0, 0)) = d((0, 0), (1, 0)) = 1 = d(A, B)
\]

If \((0, x) \in A\) is a best proximity point for \(T\), then
\[
1 = d(A, B) = d((0, x), T(0, x)) = d((0, x), (1, \frac{x}{1 + x})) = \sqrt{1 + (x - \frac{x}{1 + x})^2},
\]
and this gives us
\[
(x - \frac{x}{1 + x}) = 0
\]
the solution of (12) is \(x = 0\) and is unique. Hence \((0, 0) \in A\) is unique best proximity point for \(T\).
References


