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Research paper

On double stage minimax-shrinkage estimator for generalized Rayleigh model

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Abstract

This paper is concerned with minimax shrinkage estimator using double stage shrinkage technique for lowering the mean squared error, intended for estimate the shape parameter (α) of Generalized Rayleigh distribution in a region (R) around available prior knowledge (α_0) about the actual value (α) as initial estimate in case when the scale parameter (λ) is known .

In situation where the experimentations are time consuming or very costly, a double stage procedure can be used to reduce the expected sample size needed to obtain the estimator.

The proposed estimator is shown to have smaller mean squared error for certain choice of the shrinkage weight factor $\psi(\cdot)$ and suitable region R.

Expressions for Bias, Mean squared error (MSE), Expected sample size [E (n/α, R)], Expected sample size proportion [E(n/α,R)/n], probability for avoiding the second sample $[p(\hat{\alpha}_1 \in R)]$ and percentage of overall sample saved $[\frac{n_2}{n} p(\hat{\alpha}_1 \in R) * 100]$ for the proposed estimator are derived.

Numerical results and conclusions for the expressions mentioned above were displayed when the consider estimator are testimator of level of significanceΔ.

Comparisons with the minimax estimator and with the most recent studies were made to shown the effectiveness of the proposed estimator.

Keywords: *Generalized Rayleigh Distribution, Maximum Likelihood Estimator, Minimax Estimator, Double Stage Shrinkage Estimator, Mean Squared Error and Relative Efficiency.*

1. Introduction

Burr [4] introduced different forms of cumulative distribution functions for modeling lifetime data. Among those distributions, Burr Type X and Burr Type XII are the most popular ones. Several authors considered different aspects of the Burr Type X and Burr Type XII distribution, see for example [12], [14] and [19]. Also, Burr Type X has been studied by [3] and [13].

Two parameters Burr Type X distribution and correctly named as two parameter Generalized Rayleigh distribution are introduced in [15] and [16]. They showed that the two parameters Generalized Rayleigh distribution can be used quite effectively in modeling strength data and also in modeling general life time data. Different estimators are considered in [7] and they studied how the estimator of the different unknown parameter behaves for different sample size.

The two parameters Generalized Rayleigh (GR) distribution has the following distribution function:

$$F(x;\alpha,\lambda) = [1-e^{-(\lambda x)^2}]^{\alpha} \quad \text{for } x > 0,\ \alpha > 0,\ \lambda > 0 \qquad (1)$$

Thus, the probability density function (p.d.f.) of (GR) distribution is

$$f(x;\alpha,\lambda) = \begin{cases} 2\alpha\lambda^2 x\, e^{-(\lambda x)^2} (1-e^{-(\lambda x)^2})^{\alpha-1} & \text{for } x > 0, \alpha, \lambda > 0 \\ 0 & \text{o.w.} \end{cases} \qquad (2)$$

Where, α and λ are the shape and scale parameters respectively.

A double stage shrinkage estimator procedure has the following steps:

Let x_{1i}; i = 1, 2… n_1 be a random sample of n_1 from GR distribution and $\hat{\alpha}_1$ be a "good" estimator of α based on these n_1 observation. Construct a preliminary test region R in the parameter space α based on α_0 and an appropriate criterion for test statistic.

If $\hat{\alpha}_1 \in$ R, shrink $\hat{\alpha}_1$ towards α_0 by shrinkage weight factor $0 \le \psi(\hat{\alpha}) \le 1$ and use the shrinkage estimator $\psi(\hat{\alpha}_1)\hat{\alpha}_1 + (1-\psi(\hat{\alpha}_1))\alpha_0$, for estimate α.

If $\hat{\alpha}_1 \notin$R, obtain x_{2i}; i = 1, 2… n_2, an additional sample of size n_2 and use a pooled estimator $\hat{\alpha}_p$ of α based on combined samples of size n = n_1 + n_2, i.e.; $\hat{\alpha}_p = \frac{n_1\hat{\alpha}_1 + n_2\hat{\alpha}_2}{n}$.

Thus, the double stage shrinkage estimator (DSSE) of α will be:

$$\tilde{\alpha}_{DS} = \begin{cases} \Psi_1(\hat{\alpha}_1)\hat{\alpha}_1 + (1-\Psi_1(\hat{\alpha}_1))\alpha_0, & \text{if } \hat{\alpha}_1 \in R \\ \hat{\alpha}_p, & \text{if } \hat{\alpha}_1 \notin R \end{cases} \qquad (3)$$

The motivation of this study was provided by the work of [1], [2], [6], [9] and [18].
The minimax estimation is an upgraded non-classical approach in the estimated area of statistical inference, which was introduced by Abraham Wald (1945) from the concept of game theory. It opens a new dimension in statistical estimation and enriched the method of point estimations. Von Neumann [1944] introduced the word minimax in game theory which is the optimum strategy of the second player in the two person zero game. According to Abraham Wald, "minimax approach tries to guard against the worst by requiring that the chosen decision rule should provide maximum protection against the highest possible risk". An estimator having this property is called a minimax estimator. The most important elements in the minimax approach are the specification of the prior distribution and the loss function used. In this paper, modified linear exponential (MLINEX) loss functions have been used to obtain the minimax estimators of the parameter of the Generalized Rayleigh distribution, [5], [8], [10] and [11].
The aim of this paper is to create the double stage shrinkage estimator (DSSE) $\tilde{\alpha}_{DS}$ defined by (3) during employ the minimax estimator ($\hat{\alpha}_{B1} = \hat{\alpha}_{MML}$) instead of the classical estimator (MLE) $\hat{\alpha}_1$ for estimate the shape parameter (α) of two parameters Generalized Rayleigh (GR) distribution when the scale parameter (λ) is known.
The expressions of Bias, Mean squared error (MSE), Relative Efficiency [R.Eff(·)], Expected sample size, Expected sample size proportion, probability for avoiding the second sample and percentage of overall sample saved are derived and obtained for the proposed estimator $\tilde{\alpha}_{DS}$.
Numerical results and conclusions due mentioned expressions including some constants are performed and displayed in annexed tables.
Comparisons between the proposed estimator with the minimax estimator $\hat{\alpha}_{B1}$ and with some of the last studies are demonstrated.

2. Minimax shrinkage estimator of α

In this section, we obtain the minimax estimators of the parameter α for the Burr-X distribution. The derivation depends primarily on a theorem, which is due to Hodge and Lehmann (1950), [5], [8], [10] and [11], and can be stated as follows:
Lehmann's Theorem: If $\tau = \{F_\alpha; \alpha \in \Theta\}$ be a family of distribution functions and D a class of estimators of α. Suppose that $d^* \in D$ is a Bayes estimator against a prior distribution $\xi^*(\alpha)$ on the parameter space Θ , and the risk function $R(d^*, \alpha)$ constant on Θ ; then d^* is a minimax estimator of α.

3. Main results

Theorem: *Let* $x_1, x_2, \ldots, x_n$ *be n independently and identically distributed random variables drawn from the density (2), then* $d^* = \hat{\alpha}_{MML} = \left[\frac{\Gamma(n)}{\Gamma(n-c)}\right]^{\frac{1}{c}} / \sum_{i=1}^{n} ln\left(1 - e^{-(\lambda x_i)^2}\right)^{-1}$ *is the minimax estimator* $\hat{\alpha}_{MML}$ *of the parameter α for the MLINEX loss function of the type,*

$$L(\alpha, d_2) = \omega[\left(\frac{\hat{\alpha}_B}{\alpha}\right)^c - c\ln(\frac{\hat{\alpha}_B}{\alpha}) - 1] \,;\; \omega > 0\,, c \neq 0 \quad (4)$$

Where d^* is the estimate of α, ω and c are two known parameters the loss function.
Proof: to prove the theorem we have to use Lehmann's theorem, in order to prove the theorem it will be sufficient to show that $d^* = \hat{\alpha}_B = \left[\frac{\Gamma(n)}{\Gamma(n-c)}\right]^{\frac{1}{c}} \frac{1}{\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right)^{-1}}$ is a minimax estimator $\hat{\alpha}_{MML}$ of α for the loss function (4). For this, first we have to find the Bayes estimator d^* of α. Then if we can show that the risk of d^* is constant, then the theorem will be proved. As before, using the non-informative prior, we get the posterior distribution of α as:

$$g(\alpha|x_1, x_2, \ldots, x_n) = \frac{[\prod_{i=1}^{n} f(x;\alpha)]\, g(\alpha)}{\int_\alpha [\prod_{i=1}^{n} f(x;\alpha)]\, g(\alpha) d\alpha}$$

$$\therefore g(\alpha|x_1, x_2, \ldots x_n) = \frac{\alpha^{n-1}}{\Gamma(n)} \left(\sum_{i=1}^{n} \ln\left(1 - e^{-(\lambda x_i)^2}\right)^{-1}\right)^n e^{-\alpha \sum_{i=1}^{n} \ln\left(1-e^{-(\lambda x_i)^2}\right)^{-1}} \;;\; \alpha >, \lambda > 0\,, x > 0 \quad (5)$$

Now the Bayes estimator of α under the MLINEX loss function (4) is:

$$d^* = \hat{\alpha}_{BML} = [E_\alpha(\alpha^{-c})]^{-\frac{1}{c}}$$

Where

$$E_\alpha(\alpha^{-c}) = \int_\alpha \alpha^{-c} g\,(\alpha|x) d\alpha$$

$$= \frac{\left(\sum_{i=1}^{n}\left(1-e^{-(\lambda x_i)^2}\right)^{-1}\right)^n}{\Gamma(n)} \int_0^\infty \alpha^{-c} e^{-\alpha\left(1-e^{-(\lambda x_i)^2}\right)^{-1}} \alpha^{n-1}\, d\alpha$$

$$= \frac{\left(\sum_{i=1}^{n}\left(1-e^{-(\lambda x_i)^2}\right)^{-1}\right)^n}{\Gamma(n)} \cdot \frac{\Gamma(n-c)}{\left(\sum_{i=1}^{n}\left(1-e^{-(\lambda x_i)^2}\right)^{-1}\right)^{n-c}}$$

$$= \frac{\Gamma(n-c)}{\Gamma(n)} \cdot \left(\sum_{i=1}^{n}(1 - e^{-(\lambda x_i)^2})^{-1}\right)^c$$

Using this result we get

$$d^* = \hat{\alpha}_{BML} = \left[\frac{\Gamma(n-c)}{\Gamma(n)} \cdot \left(\sum_{i=1}^{n}\left(1 - e^{-(\lambda x_i)^2}\right)^{-1}\right)^c\right]^{-1/c} = \left[\frac{\Gamma(n-c)}{\Gamma(n)}\right]^{-1/c} \frac{1}{\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right)^{-1}} = \frac{K}{T}$$

Where $K=\left[\frac{\Gamma(n-c)}{\Gamma(n)}\right]^{-1/c}$ and $T=\ln \sum_{i=1}^{n}\left(1 - e^{-(\lambda x_i)^2}\right)^{-1}$ is the complete sufficient statistic for α.
Now the risk function under the MLINEX (4) is given by

$$R_M(\alpha) = E[L(\alpha, \hat{\alpha}_{BML})]$$

$$= \omega E[\left(\frac{\hat{\alpha}_{BML}}{\alpha}\right)^c - c ln\left(\frac{\hat{\alpha}_{BML}}{\alpha}\right) - 1$$

$$= \omega[\frac{1}{\alpha^c} E(\hat{\alpha}_{BML}{}^c) - cE(ln\, \hat{\alpha}_{BML}) + c ln\, \alpha - 1]$$

Hence we get,

$$E(\hat{\alpha}_{BML}{}^c) = E[\left(\frac{K}{T}\right)^c] = K^c E[(T^{-c})]$$

And

$$E(T^{-c}) = \int_t t^{-c}\, h(t) dt$$

$$;\, h(t)\ is\ function\ of\ gamm\ distribution$$

$$= \frac{\alpha^n}{\Gamma(n)} \int_0^\infty e^{-\alpha t}\, t^{(n-c)-1} dt$$

$$= \frac{\alpha^n \Gamma(n-c)}{\Gamma(n)\alpha^{n-c}} = \frac{\alpha^c \Gamma(n-c)}{\Gamma(n)}.$$

Using this result we get

$$E(\hat{\alpha}_{BML}{}^c) = \frac{K^c \alpha^c \Gamma(n-c)}{\Gamma(n)} = \left\{\left(\frac{\Gamma(n)}{\Gamma(n-c)}\right)^{\frac{1}{c}}\right\}^c \frac{\alpha^c \Gamma(n-c)}{\Gamma(n)} = \alpha^c$$

And

$E(\ln\hat{\alpha}_{BML}) = E[\ln\left(\frac{K}{T}\right)] = \ln K - E(\ln T)$

Here,

$E(\ln T) = \frac{\alpha^n}{\Gamma(n)}\int_0^\infty \ln t\, e^{-\alpha t} t^{n-1} dt.$

Using the relation $\alpha t = y \Rightarrow t = \frac{1}{\alpha} y$

$\therefore dt = \frac{1}{\alpha} dy$

We get

$$E(\ln T) = \frac{\alpha^n}{\Gamma(n)}\int_0^\infty \ln\left(\frac{1}{\alpha}y\right) e^{-y}\left(\frac{1}{\alpha}y\right)^{n-1}\frac{1}{\alpha}\,dy$$

$$= \frac{-\ln\alpha}{\Gamma(n)}\int_0^\infty e^{-y} y^{n-1} dy + \frac{1}{\Gamma(n)}\int_0^\infty \ln y\; e^{-y} y^{n-1} dy$$

$$= \frac{-\ln\alpha}{\Gamma(n)}\Gamma(n) + \frac{\Gamma'(n)}{\Gamma(n)}$$

$$= -\ln\alpha + \frac{\Gamma'(n)}{\Gamma(n)}$$

Where $\Gamma'(n) = \int_0^\infty \ln y\; e^{-y} y^{n-1} dy$ is the first derivative of $\Gamma(n)$ with respect to n.
Using these results we get

$$E(\ln\hat{\alpha}_{BML}) = \ln K + \ln\alpha - \frac{\Gamma'(n)}{\Gamma(n)}.$$

Now the risk function becomes

$$R_M(\alpha) = \omega\left[\frac{\alpha^c}{\alpha^c} + c\ln\alpha - c\ln\alpha - c\ln K + \frac{c\Gamma'(n)}{\Gamma(n)} - 1\right]$$

$$= \omega\left[\ln K^{-c} + \frac{c\Gamma'(n)}{\Gamma(n)}\right] \quad (6)$$

Which is constant w.r.t α, as c and n are known and independent on.
So from Lehmann's theorem it follows that $d^* = \hat{\alpha}_B = \hat{\alpha}_{MML} = \left[\frac{\Gamma(n)}{\Gamma(n-c)}\right]^{\frac{1}{c}} \frac{1}{\sum_{i=1}^n \ln\left(1-e^{-(\lambda x_i)^2}\right)^{-1}}$ is the minimax estimator of the parameter α of the Generalized Rayleigh distribution under the MLINEX loss function (4)
As well-known, the maximum likelihood estimator (MLE) for the shape parameter of two parameter GR (α,λ) when $\lambda = 1$ (λ *is konwn*), is

$$\hat{\alpha}_{mle} = -\frac{n}{\sum_{i=1}^n \ln\left(1-e^{-x_i^2}\right)} \quad (7)$$

Note that, if $x_i \sim$ GR (α,1), then $-\alpha\sum_{i=1}^{n}\ln(1-e^{-x_i^2})$ follows Gamma distribution with shape parameter (n) and scale parameter 1; G(n,1).

i.e.; $E(\hat{\alpha}_{mle}) = \frac{n}{n-1}\alpha$ and $var(\hat{\alpha}_{mle}) = \frac{n^2}{(n-1)^2(n-2)}\alpha^2$.

By using (7),

$$let\ \hat{\alpha}_B = \frac{K}{n}.\hat{\alpha}_{mle} = \frac{K}{T};\ T = -\alpha\sum \ln\left(1-e^{-x^2}\right) \sim G(n,1), \quad (8)$$

Then

$E(\hat{\alpha}_B) = \frac{K}{n}E(\hat{\alpha}_{mle}) = \frac{K}{(n-1)}\alpha$

$\therefore Bias(\hat{\alpha}_B) = E(\hat{\alpha}_B) - \alpha = \frac{K\alpha - \alpha(n-1)}{n-1}$

$Var(\hat{\alpha}_B) = \left(\frac{K}{n}\right)^2 . Var(\hat{\alpha}_{mle}) = \frac{K^2}{(n-1)^2(n-2)}\alpha^2$

$$\therefore MSE(\hat{\alpha}_B) = Var(\hat{\alpha}_B) + [Bias(\hat{\alpha}_B)]^2 = \frac{K^2}{(n-1)^2(n-2)}\alpha^2 + \left(\frac{K\alpha - \alpha(n-1)}{n-1}\right)^2$$

$$= \frac{K^2 + K^2(n-2)\alpha^2 - 2K(n-1)(n-2)\alpha^2 + (n-2)(n-1)^2\alpha^2}{(n-1)^2(n-2)}$$

4. Double stage shrinkage estimator (DSSE) $\tilde{\alpha}$

In this section, we consider the (DSSE) $\tilde{\alpha}_{DS}$ which is defined in (3) using $\hat{\alpha}_B$ defined by (8), when $\Psi(\hat{\alpha}_1) = h = \frac{k\alpha_0}{\hat{\alpha}\, Z}$; $Z = X^2{}_{\left(1-\frac{\Delta}{2},2n\right)}$ & $k = \left[\frac{\Gamma(n)}{\Gamma(n-c)}\right]^{\frac{1}{c}}$, such that (0 < h < 1) for estimate the shape parameter α of GR distribution when λ = 1

$$\tilde{\alpha}_{DS} = \begin{cases} h\hat{\alpha}_{B1} + (1-h)\alpha_0\ , if\ \hat{\alpha}_{B1} \in R \\ \hat{\alpha}_p = \frac{n1\hat{\alpha}_{B1} + n2\hat{\alpha}_{B2}}{n}\ , if\ \hat{\alpha}_{B1} \notin R \end{cases} \quad (9)$$

Where $\hat{\alpha}_{B1}$ is the minimax estimator of α in the sample $(ni); i = 1,2.\ i.e. \hat{\alpha}_{B1} = \left[\frac{\Gamma(ni)}{\Gamma(ni-c)}\right]^{\frac{1}{c}} \frac{1}{\sum_{j=1}^{ni}\ln\left(1-e^{-(\lambda x_j)^2}\right)^{-1}}$, and R is a pretest region for testing the hypothesis $H_0: \alpha = \alpha_0$ vs. $H_A = \alpha \neq \alpha_0$ with level of significance (Δ) using test statistic function $TT = \frac{2k\alpha_0}{\hat{\alpha}_{B1}}$

$$\text{i.e.;} R = \left[a < \frac{2k\alpha_0}{\hat{\alpha}_{B1}} < b\right] \quad (10)$$

Where $a = X^2{}_{(1-\Delta,2n1)}$ and $b = X^2{}_{(\Delta,2n1)}$

Are respectively the lower and upper 100(Δ/2) percentile point of chi-square distribution with degree of freedom ($2n_1$).
The expression for Bias of DSSE ($\tilde{\alpha}_{DS}$) is defined as below

$$Bias(\tilde{\alpha}_{DS}|\alpha;R) = E(\tilde{\alpha}_{DS} - \alpha) = \int_{\hat{\alpha}_{B2}=0}^\infty\int_{\hat{\alpha}_{B1}\in R}[h(\hat{\alpha}_{B1} - \alpha_0) + (\alpha_0 - \alpha)]f(\hat{\alpha}_{B1})f(\hat{\alpha}_{B2})d\hat{\alpha}_{B1}d\hat{\alpha}_{B2} + \int_{\hat{\alpha}_{B2}=0}^\infty\int_{\hat{\alpha}_{B1}\in\bar{R}}(\hat{\alpha}_p - \alpha)f(\hat{\alpha}_{B1})f(\hat{\alpha}_{B2})d\hat{\alpha}_{B1}d\hat{\alpha}_{B2}$$

Where $\bar{R}$ is the complement region of R in real space and $\hat{\alpha}_{Bi} \sim IG(n, k\alpha)$ such that:

$$f(\hat{\alpha}_{Bi}) = \frac{\left(\frac{k\alpha}{\hat{\alpha}_{Bi}}\right)^{n+1} . e^{-\frac{k\alpha}{\hat{\alpha}_{Bi}}}}{\Gamma(n).k\alpha}\quad \hat{\alpha}_{Bi} > 0 \quad (11)$$

We conclude,

$$Bias(\tilde{\alpha}_{DS}|\alpha;R) = \alpha\left\{\frac{k}{c}\zeta J_0(a^*,b^*) - \frac{1}{c}\zeta^2 J'_1(a^*,b^*) + (\zeta-1)J_0(a^*,b^*) + \left(\frac{1}{1+u}\right)\left[\frac{k-(n1-1)}{(n1-1)}\right] + \left(\frac{u}{1+u}\right)\left[\frac{k-(n2-1)}{(n2-1)}\right] - \left(\frac{1}{1+u}\right)[kJ_1(a^*,b^*) - J_0(a^*,b^*)] - \left(\frac{u}{1+u}\right)\left[\frac{k-(n2-1)}{(n2-1)}\right]J_0(a^*,b^*)\right\}$$

$$\text{Where } J_\ell(a^*,b^*) = \int_{a^*}^{b^*} y^{-\ell}\frac{y^{n-1}e^{-y}}{\Gamma(n)}dy;\ \ell = 0,1,2 \quad (12)$$

Also

$$\zeta = \frac{\alpha_0}{\alpha}\ , a^* = (2\zeta)^{-1}.a\ , b^* = (2\zeta)^{-1}.b\ , u = \frac{n2}{n1}\ and\ y = \left({}^{k}/_{\hat{\alpha}_{B1}}\right)\alpha \quad (13)$$

The Bias ratio [B ($\cdot$)] of DSSE ($\tilde{\alpha}_{DS}$) is defined as:

$$B(\tilde{\alpha}_{DS}) = \frac{Bias(\tilde{\alpha}_{DS}|\alpha;R)}{\alpha} \quad (14)$$

The expression of Mean squared error [MSE($\cdot$)] of $\hat{\alpha}_{DS}$ derived as below:-

$MSE(\tilde{\alpha}_{DS}|\alpha;R) = E(\tilde{\alpha}_{DS}-\alpha)^2 = \int_{\hat{\alpha}_{B2}=0}^{\infty}\int_{\hat{\alpha}_{B1}\in R}[h(\hat{\alpha}_{B1}-\alpha_0)+(\alpha_0-\alpha)]^2 f(\hat{\alpha}_{B1})f(\hat{\alpha}_{B2})d\hat{\alpha}_{B1}d\hat{\alpha}_{B2} + \int_{\hat{\alpha}_{B2}=0}^{\infty}\int_{\hat{\alpha}_{B1}\in\bar{R}}(\hat{\alpha}_p-\alpha)^2 f(\hat{\alpha}_{B1})f(\hat{\alpha}_{B2})d\hat{\alpha}_{B1}d\hat{\alpha}_{B2}$

And by simple computations, one can get:

$$MSE(\tilde{\alpha}_{DS}|\alpha;R) = \alpha^2\Big\{\frac{k^2}{c^2}\zeta^2 J_0(a^*,b^*) - 2\frac{k}{c^2}\zeta^3 J'_1(a^*,b^*) + \frac{1}{c^2}\zeta^4 J'_2(a^*,b^*) + 2\frac{k}{c}\zeta(\zeta-1)J_0(a^*,b^*) - 2\frac{1}{c}\zeta^2(\zeta-1)J'_1(a^*,b^*) + (\zeta-1)^2 J_0(a^*,b^*) + \left(\frac{1}{1+u}\right)^2\left[\frac{k^2+k^2(n1-2)-2k(n1-1)(n1-2)+(n1-2)(n1-1)^2}{(n1-2)(n1-1)^2}\right] + 2\left(\frac{u}{(1+u)^2}\right)\left[\frac{k-(n1-1)}{(n1-1)}\right]\left[\frac{k-(n2-1)}{(n2-1)}\right] + \left(\frac{u}{1+u}\right)^2\left[\frac{k^2+k^2(n2-2)-2k(n2-1)(n2-2)+(n2-2)(n2-1)^2}{(n2-2)(n2-1)^2}\right] - \left(\frac{1}{1+u}\right)^2[k^2J_1(a^*,b^*) - 2kJ_1(a^*,b^*) + J_0(a^*,b^*)] - 2\left(\frac{u}{(1+u)^2}\right)\left[\frac{k-(n2-1)}{(n2-1)}\right][kJ_1(a^*,b^*) - J_0(a^*,b^*)] - \left(\frac{u}{1+u}\right)^2\left[\frac{k^2+k^2(n2-2)-2k(n2-1)(n2-2)+(n2-2)(n2-1)^2}{(n2-2)(n2-1)^2}\right]J_0(a^*,b^*)\Big\}$$

Now, the Efficiency of $\tilde{\alpha}_{DS}$ relative to $\hat{\alpha}_{B1}$ which is denoted by $R.Eff(\tilde{\alpha}_{DS}|\alpha;R)$ is defined by:

$$R.Eff(\tilde{\alpha}_{DS}|\alpha;R) = \frac{MSE(\hat{\alpha}_{B1})}{MSE(\tilde{\alpha}_{DS}|\alpha;R)\cdot[E(n/\alpha,R)/n]} \qquad (15)$$

Where E (n/α,R) is the Expected sample size, which is defined as:

$$E(n/\alpha,R) = n\left[1-\frac{u}{1+u}J_0(a^*,b^*)\right], \qquad (16)$$

As well as the Expected sample size proportion E (n/α,R)/n equal to

$$1-\frac{u}{1+u}J_0(a^*,b^*) \qquad (17)$$

Also, we have to define the percentage of the overall sample saved (p.o.s.s.) of $\tilde{\alpha}_{DS}$ as:

$$p.o.s.s. = \frac{n_2}{n}J_0(a^*,b^*)*100 \qquad (18)$$

And, finally, $p(\hat{\alpha}_1 \in R)$ represent the probability of a voiding the second sample (stage).
See for example [1], [2]. [6], [9], and [17].

5. Discussion and numerical results

The computations of Relative Efficiency [R.Eff (·)] and Bias Ratio [B (·)], Expected sample size [E (n/α,R)], Expected sample size proportion [E(n/α,R)/n], Percentage of the overall sample saved (p.o.s.s.) and probability of a voiding the second sample [$p(\hat{\alpha}_1 \in R)$] were used for the estimator $\tilde{\alpha}_{DS}$. These computations were performed using Mathcad program for n_1 = 4,6,8,10,12, u(= n_2/n_1) =2,6,8,10,12, $\zeta = \alpha_0/\alpha$ = 0.25(0.25)2, Δ = 0.01,0.05,0.1,c=2 and h $= \frac{k\alpha_0}{\tilde{\alpha}\, Z}$.

By using Mathcad program some of these computations are given in the tables (1)-(13).

The observation mentioned in the tables lead to the following results:

i) The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}_{DS}$ are adversely proportional with small value of Δ especially when ζ = 1, i.e. Δ = 0.01 yield highest efficiency (see Tables (1), (5) and (9)).

ii) The Relative Efficiency [R.Eff (·)] of $\tilde{\alpha}_{DS}$ has maximum value when $\alpha = \alpha_0$ (ζ =1), for each n_1,Δ and decreasing otherwise ($\zeta \neq 1$). This feature shown the important usefulness of prior knowledge which given higher effects of proposed estimator as well as the important role of shrinkage technique and its philosophy (see Tables (1), (5) and (9)).

iii) Bias ratio [B (·)] of $\tilde{\alpha}_{DS}$ are reasonably small when $\alpha = \alpha_0$ for each n_1, and same Δ, and increases otherwise. This property shown that the proposed estimator $\tilde{\alpha}_{DS}$ is very closely to unbiasedness property especially when $\alpha = \alpha_0$ (see Tables (1), (5) and (9)).

iv) The Effective interval of $\tilde{\alpha}_{DS}$ [the value of $\tilde{\alpha}_{DS}$ which makes R.Eff(·) of $\tilde{\alpha}_{DS}$ greater than one] is approximate [0.25,2] (see Tables (1), (5) and (9)).

v) Bias ratio [B (·)] of $\tilde{\alpha}_{DS}$ are reasonably large with small value of u (see Tables (1), (5) and (9)).

vi) R.Eff ($\tilde{\alpha}_{DS}$) is decreasing function with increasing of the first sample size n_1, for each Δ and ζ (see Tables (1), (5) and (9)).

vii) The Expected value of sample size of $\tilde{\alpha}_{DS}$ is close to n_1, especially when $\zeta \cong 1$ and start faraway otherwise (see Tables (2), (6) and (10)).

viii) Percentage of the overall sample saved $\left[\frac{n_2}{n}J_0(a^*,b^*)*100\right]$ is increasing value with increasing value of u (u = n_2/n_1) and ζ (see Tables (4), (8) and (12)).

ix) R.Eff($\tilde{\alpha}_{DS}$) is an increasing function with respect to u. This property shown the effective of proposed estimator using small n_1 relative to n_2 (or large n_2) which given higher efficiency and reduce the observation cost (see Tables (1), (5) and (9)).

x) The considered estimator $\tilde{\alpha}_{DS}$ is better than the minimax estimator especially when $\alpha \approx \alpha_0$, this will give the effective of $\tilde{\alpha}_{DS}$ relative to $\hat{\alpha}_{B1}$ and also given an important weight of prior knowledge, and the augmentation of efficiency may be reach to ten times (see Tables (1), (5) and (9)).

xi) The probability of avoiding second sample of the considered estimator $\tilde{\alpha}_{DS}$ has a maximum value when $\alpha = \alpha_0$ for each $n_1$and Δ (see Tables (13)), this property shown the effective of double stage shrinkage estimators to reduce the sample size.

xii) The considered estimator $\tilde{\alpha}_{DS}$ is more efficient than the estimators introduced by [7], [9] and [11] in the sense of higher efficiency.

Table 1: Shown Bias Ratio [B (·)] and R.E.Ff of $\tilde{A}_{DS}$ W.R.T. Δ, N_1 and ζ when U =6

			ζ							
Δ	n_1	R.Eff(-) B(-)	0.25	0.50	0.75	1	1.25	1.50	1.75	2
0.01	4	R.Eff(-)	3.8375436	17.0374954	70.8568096	349.5356128	54.9780952	14.0415248	5.4920024	2.7364843
		B(-)	-0.7361405	-0.501523	-0.501523	-6.033328E-3	0.2202785	0.4065392	0.5400779	0.6198681
	6	R.Eff(-)	2.2357931	13.2897009	67.9836612	446.9918537	52.0188327	12.0565522	4.4887373	2.2291462
		B(-)	-0.7372543	-0.5103292	-0.2546055	-6.621779E-3	0.2142392	0.3754864	0.45856	0.4679973
	8	R.Eff(-)	1.6795106	10.2072117	64.3156313	516.2703674	49.0336014	10.5401112	3.8171404	1.9339236
		B(-)	-0.7386806	-0.5222446	-0.2572666	-6.849207E-3	0.2083544	0.3433069	0.3757137	0.3212556
	10	R.Eff(-)	1.481969	7.8517441	60.3960552	568.5749596	46.1709369	9.3115594	3.3327567	1.747586
		B(-)	-0.7380109	-0.5368034	-0.2600615	-6.959501E-3	0.2023992	0.3100801	0.293013	0.1833459

	12	R.Eff(-)	1.4170837	6.1274647	56.4217269	609.5683911	43.461959	8.2932353	2.9697037	1.6255451
		B(-)	-0.7358799	-0.5532719	-0.2630556	-7.021246E-3	0.1963193	0.2760166	0.2117398	0.0565818
0.05	4	R.Eff(-)	2.0367028	9.4204389	45.3879879	121.737409	28.4121193	8.3573908	3.6165638	2.0052434
		B(-)	-0.7560093	-0.5288025	-0.2706274	-0.0339792	0.1509111	0.2681358	0.321622	0.3255413
	6	R.Eff(-)	1.531913	6.4309968	39.4013805	132.4978401	25.7079751	6.9676297	2.9899699	1.7201642
		B(-)	-0.7526109	-0.5520324	-0.2786634	-0.0346019	0.1376973	0.2142247	0.2063821	0.1428628
	8	R.Eff(-)	1.4063866	4.6773735	34.0398646	137.8569218	23.3723771	5.9898986	2.6003903	1.5721905
		B(-)	-0.7470369	-0.5768396	-0.2869036	-0.0349426	0.1248292	0.1619428	0.1000978	-0.0140012
	10	R.Eff(-)	1.3830997	3.6073693	29.5192187	141.123789	21.3533649	5.2537323	2.3339035	1.4890989
		B(-)	-0.7414922	-0.6009981	-0.2952873	-0.0351368	0.1121118	0.1112742	2.8841824E-3	-0.1467708
	12	R.Eff(-)	1.3850913	2.9271336	25.7390545	143.3368334	19.5935066	4.6787938	2.1416783	1.4416449
		B(-)	-0.7370681	-0.623249	-0.3038255	-0.0352589	0.0994719	0.0623025	-0.0853298	-0.2577666
0.1	4	R.Eff(-)	1.6314999	6.1987327	28.3156034	55.2808856	17.4312967	6.0554446	2.9128384	1.7531484
		B(-)	-0.769969	-0.5598562	-0.297209	-0.0712784	0.0818908	0.1581071	0.1732989	0.1477248
	6	R.Eff(-)	1.4051012	4.2510831	23.8450315	58.2022606	15.7526535	5.0976112	2.4700569	1.5607989
		B(-)	-0.7581989	-0.588899	-0.3085423	-0.0710792	0.0654675	0.0957881	0.0509138	-0.0331327
	8	R.Eff(-)	1.369939	3.1978839	20.119457	59.4843514	14.3148601	4.4292617	2.1982133	1.4683349
		B(-)	-0.7487252	-0.6165186	-0.3204947	-0.0711455	0.0491657	0.0368678	-0.0570746	-0.1798678
	10	R.Eff(-)	1.3741103	2.5822991	17.1643208	60.2163193	13.0867277	3.9307041	2.0143657	1.421593
		B(-)	-0.7419052	-0.6405443	-0.3325064	-0.0712066	0.0331166	-0.0186425	-0.1517676	-0.2977595
	12	R.Eff(-)	1.3831841	2.2003462	14.8128696	60.6927863	12.0277307	3.5442343	1.8831179	1.3987802
		B(-)	-0.7371543	-0.6604705	-0.3444782	-0.071251	0.0173067	-0.0708363	-0.2343807	-0.3917032

Table 2: Shown Expected Sample Size of $\tilde{\alpha}$ W.R.T. Δ, U, and ζ U=6

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.50	1.25	1.75	2
		0.01	10.798	5.143	4.325	4.240	4.648	5.627	7.099	8.881
	4	0.05	19.205	8.242	5.508	5.200	6.236	8.131	10.419	12.752
		0.1	23.066	11.040	6.888	6.401	7.822	10.147	12.700	15.117
		0.01	26.800	9.296	6.669	6.360	7.175	9.329	12.617	16.490
	6	0.05	37.397	16.108	8.862	7.800	9.824	13.789	18.493	23.051
		0.1	40.135	21.252	11.247	9.600	12.355	17.108	22.158	26.640
		0.01	46.605	15.134	9.160	8.480	9.844	13.681	19.512	26.099
6	8	0.05	54.321	26.604	12.647	10.400	13.725	20.457	28.199	35.250
		0.1	55.498	34.003	16.225	12.801	17.289	25.188	33.207	39.823
		0.01	65.545	22.965	11.820	10.600	12.661	18.713	27.761	37.483
	10	0.05	69.521	39.398	16.878	13.000	17.935	28.089	39.309	48.859
		0.1	69.892	48.586	21.812	16.001	22.616	34.290	45.532	54.116
		0.01	82.277	32.904	14.665	12.720	15.634	24.447	37.308	50.366
	12	0.05	83.886	54.006	21.566	15.600	22.458	36.636	51.611	63.466
		0.1	83.980	64.339	28.000	19.199	28.327	44.315	58.859	69.103

Table 3: Shown Expected Sample Size Proportion W.R.T. Δ, U, N_1 and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.25	1.50	1.75	2
		0.01	0.386	0.184	0.154	0.151424	0.166	0.201	0.254	0.317
	4	0.05	0.686	0.294	0.197	0.185719	0.223	0.290	0.372	0.455
		0.1	0.824	0.394	0.246	0.22859	0.279	0.362	0.454	0.540
		0.01	0.638	0.221	0.159	0.15142	0.171	0.222	0.300	0.393
	6	0.05	0.890	0.384	0.211	0.185716	0.234	0.328	0.440	0.549
		0.1	0.956	0.506	0.268	0.22857	0.294	0.407	0.528	0.634
		0.01	0.832	0.270	0.164	0.15143	0.176	0.244	0.348	0.466
6	8	0.05	0.970	0.475	0.226	0.1857216	0.245	0.365	0.504	0.629
		0.1	0.991	0.607	0.290	0.22858	0.309	0.45	0.593	0.711
		0.01	0.936	0.328	0.169	0.151	0.181	0.267	0.397	0.535
	10	0.05	0.993	0.563	0.241	0.1857	0.256	0.401	0.562	0.698
		0.1	0.998	0.694	0.312	0.22857	0.323	0.490	0.65	0.773
		0.01	0.979	0.392	0.175	0.151	0.186	0.291	0.444	0.600
	12	0.05	0.999	0.643	0.257	0.186	0.267	0.436	0.614	0.756
		0.1	1	0.766	0.333	0.229	0.337	0.528	0.701	0.823

Table 4: Shown Percentage of Overall Sample Saved W.R.T. Δ, U, N_1 and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.25	1.50	1.75	2
		0.01	61.435	81.633	84.553	84.858	83.399	79.904	74.646	68.281
6	4	0.05	31.411	70.564	80.329	81.4281	77.729	70.961	62.790	54.456
		0.1	17.621	60.571	75.400	77.140	72.065	63.762	54.642	46.011

	0.01	36.189	77.866	84.122	84.8571	82.917	77.788	69.959	60.739
6	0.05	10.959	61.648	78.900	81.4284	76.609	67.170	55.968	45.118
	0.1	4.440	49.400	73.222	77.143	70.584	59.267	47.242	36.571
	0.01	16.776	72.976	83.642	84.85722	82.422	75.569	65.158	53.394
8	0.05	2.998	52.493	77.416	81.427	75.491	63.469	49.645	37.054
	0.1	0.897	39.28	71.027	77.141	69.128	55.021	40.703	28.888
	0.01	6.364	67.192	83.114	84.857	81.912	73.267	60.341	46.453
10	0.05	0.685	43.716	75.889	81.428	74.378	59.874	43.844	30.202
	0.1	0.155	30.591	68.84	77.142	67.692	51.015	34.954	22.691
	0.01	2.051	60.828	82.542	84.857	81.388	70.897	55.586	40.041
12	0.05	0.136	35.707	74.326	81.429	73.264	56.385	38.559	24.445
	0.1	0.024	23.406	66.667	77.144	66.278	47.245	29.929	17.735

Table 5: Shown Bias Ratio [B (·)] and R.E.Ff of $\tilde{A}_{DS}$ W.R.T. Δ, N_1 and ζ when U =10

			ζ							
Δ	n_1	R.Eff(-) B(-)	0.25	0.50	0.75	1	1.25	1.50	1.75	2
0.01	4	R.Eff(-)	4.2817412	24.837005	120.507637	877.3512371	89.423265	19.9707311	7.1365084	3.3445721
		B(-)	-0.7671927	-0.5096518	-0.254967	-7.735955E-3	0.2183015	0.402778	0.5332053	0.6089276
	6	R.Eff(-)	2.204676	17.6224172	110.1436233	973.7912909	81.3597131	16.4247816	5.5628186	2.6062566
		B(-)	-0.7948329	-0.523214	-0.2580198	-8.049341E-3	0.2119518	0.3695911	0.4465896	0.4483821
	8	R.Eff(-)	1.5503732	12.4103752	100.3133939	1.0278372E+3	74.4504247	13.8017704	4.5412861	2.1784313
		B(-)	-0.816409	-0.5411785	-0.2612029	-8.158315E-3	0.2056085	0.3350283	0.3582937	0.2928109
	10	R.Eff(-)	1.3249242	8.879597	91.0438288	1.0631064E+3	68.3282951	11.766397	3.8289052	1.9064728
		B(-)	-0.8270109	-0.5627152	-0.2646341	-8.204528E-3	0.1991337	0.299253	0.269959	0.1462593
	12	R.Eff(-)	1.2488485	6.5382455	82.3809757	1.088039E+3	62.8373031	10.1502825	3.3099691	1.7237528
		B(-)	-0.8301872	-0.5867193	-0.2683452	-8.226454E-3	0.1924972	0.2625142	0.182993	0.0112842
0.05	4	R.Eff(-)	1.9913485	11.010938	61.6130114	176.9731137	38.0323007	10.2786802	4.2098241	2.2560822
		B(-)	-0.812021	-0.5506158	-0.2804166	-0.0401446	0.1440088	0.2577245	0.3061473	0.304373
	6	R.Eff(-)	1.4041492	6.8860046	50.1591237	179.2489535	33.1088859	8.2254795	3.3451094	1.8650557
		B(-)	-0.8316223	-0.5843215	-0.2900058	-0.0404553	0.1297417	0.1993917	0.1823021	0.1090682
	8	R.Eff(-)	1.2550977	4.7161532	41.3586705	179.8703807	29.2303967	6.8299523	2.812009	1.6491903
		B(-)	-0.8364527	-0.6195409	-0.2999307	-0.0406223	0.1156854	0.142653	0.0678465	-0.0589962
	10	R.Eff(-)	1.2204993	3.4879798	34.5415686	180.1572259	26.0515016	5.8156409	2.4513157	1.5169123
		B(-)	-0.8352917	-0.6535498	-0.3101009	-0.0407132	0.1017156	0.087566	-0.0370146	-0.2015222
	12	R.Eff(-)	1.2154316	2.7473532	29.1821293	180.3258853	23.3900011	5.0485911	2.1933333	1.4309537
		B(-)	-0.8329858	-0.6847184	-0.3204927	-0.0407676	0.087791	0.0342572	-0.1322999	-0.3208826
0.1	4	R.Eff(-)	1.5422898	6.5994524	33.4943995	67.4879244	20.9961066	6.9586301	3.234285	1.9041682
		B(-)	-0.8357003	-0.5916988	-0.3130836	-0.0821233	0.0700132	0.1418965	0.1513108	0.1196385
	6	R.Eff(-)	1.2705395	4.227365	26.67804	68.0110762	18.3735949	5.6509129	2.6453922	1.6362613
		B(-)	-0.8421607	-0.6339021	-0.3271614	-0.0818935	0.0518121	0.0737521	0.0186377	-0.0753551
	8	R.Eff(-)	1.2174067	3.0469392	21.6837885	68.0738208	16.2993989	4.7645733	2.2821097	1.49085
		B(-)	-0.8397734	-0.6729395	-0.3416976	-0.0818752	0.0337825	9.326355E-3	-0.0985031	-0.2337578
	10	R.Eff(-)	1.2112723	2.3918262	17.9784345	68.0720906	14.6044467	4.1221267	2.0363483	1.4034683
		B(-)	-0.8361207	-0.7066644	-0.3562758	-0.0818747	0.0160025	-0.0514232	-0.2013373	-0.361232
	12	R.Eff(-)	1.2134814	1.9990105	15.1633208	68.0587191	13.1902204	3.6369187	1.8607347	1.3478891
		B(-)	-0.8331606	-0.7345905	-0.3707997	-0.081875	-1.533870E-3	-0.1085865	-0.2911493	-0.4629786

Table 6: Shown Expected Sample Size of $\tilde{\alpha}$ W.R.T. Δ, U, and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.50	1.25	1.75	2
10		0.01	15.331	5.905	4.542	4.400	5.081	6.711	9.165	12.135
	4	0.05	29.341	11.070	6.513	6.000	7.726	10.885	14.698	18.587
		0.1	35.777	15.733	8.813	8.001	10.370	14.244	18.500	22.528
		0.01	40.667	11.494	7.115	6.600	7.958	11.548	17.029	23.483
	6	0.05	58.329	22.846	10.770	9.000	12.374	18.981	26.822	34.418
		0.1	62.892	31.420	14.745	12.000	16.591	24.513	32.931	40.400
		0.01	72.342	19.889	9.934	8.800	11.073	17.469	27.186	38.166
	8	0.05	85.202	39.007	15.745	12.001	17.541	28.762	41.665	53.416
		0.1	87.163	51.338	21.708	16.001	23.481	36.647	50.011	61.038
		0.01	102.575	31.609	13.033	11.000	14.436	24.522	39.602	55.805
	10	0.05	109.201	58.997	21.463	15.000	23.225	40.148	58.849	74.765
		0.1	109.819	74.311	29.687	20.001	31.026	50.483	69.220	83.527
		0.01	129.129	46.840	16.442	13.200	18.056	32.745	54.180	75.943
	12	0.05	131.809	82.010	27.943	18.000	29.430	53.061	78.018	97.777
		0.1	131.967	99.232	38.666	23.999	39.211	65.858	90.099	107.171

Table 7: Shown Expected Sample Size Proportion W.R.T. Δ, U, N_1 and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.25	1.50	1.75	2
		0.01	0.348	0.134	0.103	0.099995	0.115	0.153	0.208	0.276
	4	0.05	0.667	0.252	0.148	0.136368	0.176	0.247	0.334	0.422
		0.1	0.813	0.358	0.200	0.18184	0.236	0.324	0.420	0.512
		0.01	0.616	0.174	0.108	0.1000005	0.121	0.175	0.258	0.356
	6	0.05	0.884	0.346	0.163	0.136365	0.187	0.288	0.406	0.521
		0.1	0.953	0.476	0.223	0.18182	0.251	0.371	0.499	0.612
		0.01	0.822	0.226	0.113	0.0999992	0.126	0.199	0.309	0.434
10	8	0.05	0.968	0.443	0.179	0.13637	0.199	0.327	0.473	0.607
		0.1	0.99	0.583	0.247	0.18183	0.267	0.416	0.568	0.694
		0.01	0.933	0.287	0.118	0.1000004	0.131	0.223	0.36	0.507
	10	0.05	0.993	0.536	0.195	0.136364	0.211	0.365	0.535	0.68
		0.1	0.998	0.676	0.27	0.181827	0.282	0.459	0.629	0.759
		0.01	0.978	0.355	0.125	0.099998	0.137	0.248	0.41	0.575
	12	0.05	0.999	0.621	0.212	0.136362	0.223	0.402	0.591	0.741
		0.1	1	0.752	0.293	0.18181	0.297	0.499	0.683	0.812

Table 8: Shown Percentage of Overall Sample Saved W.R.T. Δ, U, N_1 and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.25	1.50	1.75	2
		0.01	65.158	86.581	89.678	90.0004	88.453	84.747	79.170	72.420
	4	0.05	33.315	74.840	85.197	86.3631	82.44	75.261	66.595	57.757
		0.1	18.689	64.242	79.97	81.815	76.432	67.626	57.954	48.800
		0.01	38.383	82.585	89.22	89.99995	87.942	82.503	74.199	64.42
	6	0.05	11.623	65.385	83.682	86.3634	81.252	71.241	59.36	47.852
		0.1	4.709	52.394	77.659	81.8181	74.862	62.859	50.105	38.788
		0.01	17.793	77.398	88.712	90.0000	87.417	80.149	69.107	56.63
10	8	0.05	3.179	55.674	82.108	86.3629	80.067	67.316	52.654	39.3
		0.1	0.951	41.661	75.332	81.816	73.317	58.356	43.169	30.639
		0.01	6.75	71.265	88.152	89.99996	86.877	77.707	63.998	49.268
	10	0.05	0.726	46.366	80.488	86.3636	78.886	63.502	46.501	32.032
		0.1	0.164	32.445	73.012	81.8172	71.795	54.107	37.073	24.067
		0.01	2.175	64.515	87.544	90.0001	86.321	75.193	58.955	42.468
	12	0.05	0.145	37.871	78.831	86.36378	77.704	59.803	40.896	25.927
		0.1	0.025	24.825	70.708	81.8189	70.294	50.108	31.743	18.81

Table 9: Shown Bias Ratio [B (·)] and R.E.Ff of $\widetilde{A}_{DS}$ W.R.T. Δ, N_1 and ζ when U =12

			ζ							
Δ	n_1	R.Eff(-) B(-)	0.25	0.50	0.75	1	1.25	1.50	1.75	2
0.01	4	R.Eff(-) B(-)	4.4047433 -0.7754921	27.918037 -0.5118301	142.3716759 -0.2558003	1.127155E+3 -8.192209E-3	103.9902821 0.217775	22.2273411 0.40178	7.7001567 0.5313826	3.537765 0.6060257
	6	R.Eff(-) B(-)	2.1935522 -0.8102372	19.1520044 -0.5266675	128.4044828 -0.2589358	1.2011061E+3 -8.431995E-3	93.5980944 0.2113415	18.0064351 0.3680196	5.9085592 0.4433979	2.7187997 0.4431503
	8	R.Eff(-) B(-)	1.5180317 -0.8372268	13.1148455 -0.5462559	115.580423 -0.2622593	1.2403565E+3 -8.509411E-3	84.8733774 0.2048744	14.9312611 0.3328155	4.7629815 0.3536362	2.2478301 0.2852039
	10	R.Eff(-) B(-)	1.2878723 -0.8508681	9.1829104 -0.5696671	103.7460701 -0.2658617	1.2653906E+3 -8.538601E-3	77.2538604 0.1982596	12.5839884 0.2963544	3.9745636 0.2637855	1.9495403 0.1363261
	12	R.Eff(-) B(-)	1.2101468 -0.8554838	6.6520598 -0.5956964	92.8891509 -0.2697658	1.2828372E+3 -8.549971E-3	70.5045452 0.191473	10.7480868 0.2588956	3.406257 0.1752872	1.7490861 -8.60013E-4
0.05	4	R.Eff(-) B(-)	1.9760544 -0.8269652	11.4839815 -0.5564506	67.2184388 -0.2830387	194.6492471 -0.0417938	41.308214 0.1421705	10.886841 0.2549584	4.3859509 0.3020384	2.3273421 0.2987521
	6	R.Eff(-) B(-)	1.37208 -0.8527473	7.006937 -0.5929673	53.8224447 -0.2930461	194.9558908 -0.0420227	35.6154476 0.1276178	8.6091107 0.1954347	3.4455392 0.1758777	1.9039315 0.1000497
	8	R.Eff(-) B(-)	1.2192143 -0.8603953	4.723527 -0.6309848	43.8152993 -0.3034249	194.6683649 -0.0421446	31.1930226 0.1132397	7.0778917 0.1374948	2.8692324 0.0592205	1.6684975 -0.0710332
	10	R.Eff(-) B(-)	1.1828351 -0.8604337	3.4567144 -0.6676432	36.2089771 -0.3140763	194.4324999 -0.0422089	27.607124 0.0989315	5.9769631 0.081217	2.4816029 -0.0477015	1.5228587 -0.21619
	12	R.Eff(-) B(-)	1.1766267 -0.8587137	2.7030367 -0.7012118	30.3194279 -0.3249673	194.2612288 -0.0422459	24.6332576 0.0846601	5.1523752 0.0267395	2.2057159 -0.1448928	1.4270355 -0.3378066
0.1	4	R.Eff(-) B(-)	1.5175165 -0.8532268	6.6992896 -0.6002088	35.029416 -0.3173317	70.9696944 -0.0850216	22.068608 0.0668495	7.21958 0.1375871	3.3232743 0.1454687	1.9446277 0.1121761
	6	R.Eff(-) B(-)	1.2375445 -0.8646056	4.2167956 -0.6459468	27.534003 -0.3321496	71.0712865 -0.0847881	19.170121 0.0481655	5.8065903 0.0678714	2.6916435 0.0100238	1.6550344 -0.0866259

8	R.Eff(-)	1.1814512	3.0076239	22.1709488	70.9534873	16.9039944	4.8565672	2.302899	1.4953222
	B(-)	-0.8641521	-0.6880563	-0.3473829	-0.0847505	0.0296672	1.95993E-3	-0.1095857	-0.2481767
10	R.Eff(-)	1.1736085	2.3451492	18.2477612	70.8512733	15.067225	4.1735307	2.0407626	1.3976371
	B(-)	-0.8613739	-0.7243935	-0.362653	-0.0847356	0.0114186	-0.0602031	-0.2146162	-0.3782379
12	R.Eff(-)	1.1746785	1.9512073	15.2986375	70.7737677	13.5454418	3.6614247	1.8539925	1.3342339
	B(-)	-0.8589123	-0.754476	-0.3778649	-0.0847257	-6.58452E-3	-0.1187069	-0.3063706	-0.4820918

Table 10: Shown Expected Sample Size of $\tilde{A}_{DS}$ W.R.T. Δ, U, and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.50	1.25	1.75	2
		0.01	17.597	6.285	4.650	4.480	5.297	7.254	10.198	13.762
	4	0.05	34.410	12.484	7.016	6.400	8.472	12.262	16.838	21.504
		0.1	42.132	18.080	9.776	8.801	11.644	16.293	21.400	26.234
		0.01	47.601	12.592	7.338	6.720	8.350	12.658	19.234	26.979
	6	0.05	68.794	26.215	11.724	9.600	13.649	21.577	30.987	40.101
		0.1	74.270	36.504	16.494	13.200	18.709	28.215	38.317	47.280
		0.01	85.211	22.267	10.320	8.960	11.688	19.363	31.024	44.199
	8	0.05	100.643	45.208	17.294	12.801	19.450	32.914	48.398	62.499
		0.1	102.995	60.006	24.450	17.601	26.577	42.377	58.413	71.645
		0.01	121.090	35.931	13.640	11.200	15.323	27.427	45.523	64.966
	10	0.05	129.041	68.797	23.756	16.000	25.870	46.177	68.619	87.718
		0.1	129.783	87.173	33.624	22.001	35.231	58.579	81.064	98.232
		0.01	152.555	53.808	17.330	13.440	19.267	36.894	62.616	88.731
	12	0.05	155.771	96.012	31.131	19.200	32.916	61.273	91.221	114.932
12		0.1	155.960	116.678	43.999	26.399	44.654	76.629	105.719	126.205

Table 11: Shown Expected Sample Size Proportion W.R.T. Δ, U, N_1 and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.25	1.50	1.75	2
		0.01	0.338	0.121	0.089	0.086148842	0.102	0.139	0.196	0.265
	4	0.05	0.662	0.24	0.135	0.123082276	0.163	0.236	0.324	0.414
		0.1	0.81	0.348	0.188	0.169253764	0.224	0.313	0.412	0.504
		0.01	0.61	0.161	0.094	0.086154346	0.107	0.162	0.247	0.346
	6	0.05	0.882	0.336	0.15	0.123079188	0.175	0.277	0.397	0.514
		0.1	0.952	0.468	0.211	0.169230694	0.24	0.362	0.491	0.606
		0.01	0.819	0.214	0.099	0.086153046	0.112	0.186	0.298	0.425
12	8	0.05	0.968	0.435	0.166	0.123084894	0.187	0.316	0.465	0.601
		0.1	0.99	0.577	0.235	0.169243129	0.256	0.407	0.562	0.689
		0.01	0.931	0.276	0.105	0.086154292	0.118	0.211	0.35	0.5
	10	0.05	0.993	0.529	0.183	0.123077704	0.199	0.355	0.528	0.675
		0.1	0.998	0.671	0.259	0.169239908	0.271	0.451	0.624	0.756
		0.01	0.978	0.345	0.111	0.086152398	0.124	0.236	0.401	0.569
	12	0.05	0.999	0.615	0.200	0.123075521	0.211	0.393	0.585	0.737
		0.1	1	0.748	0.282	0.169222587	0.286	0.491	0.678	0.809

Table 12: Shown Percentage of Overall Sample Saved W.R.T. Δ, U, N_1 and ζ

			ζ							
u	n_1	Δ	0.25	0.50	0.75	1	1.25	1.50	1.75	2
		0.01	66.16	87.913	91.057	91.385115832	89.814	86.051	80.388	73.534
	4	0.05	33.827	75.992	86.508	87.691772394	83.708	76.419	67.619	58.645
		0.1	18.977	65.231	81.2	83.07462357	77.608	68.667	58.845	49.55
		0.01	38.973	83.856	90.593	91.384565429	89.295	83.772	75.34	65.411
	6	0.05	11.802	66.391	84.97	87.6920812	82.502	72.337	60.273	48.588
		0.1	4.782	53.2	78.854	83.076930636	76.014	63.826	50.876	39.384
		0.01	18.066	78.589	90.077	91.384695383	88.762	81.382	70.17	57.501
12	8	0.05	3.228	56.531	83.371	87.691510565	81.299	68.352	53.464	39.905
		0.1	0.966	42.302	76.49	83.075687055	74.445	59.253	43.834	31.11
		0.01	6.853	72.361	89.508	91.384570826	88.213	78.902	64.982	50.026
	10	0.05	0.738	47.079	81.726	87.692229598	80.1	64.479	47.216	32.525
		0.1	0.167	32.944	74.135	83.076009249	72.899	54.939	37.643	24.437
		0.01	2.208	65.508	88.891	91.384760162	87.649	76.35	59.862	43.121
	12	0.05	0.147	38.454	80.044	87.692447874	78.9	60.723	41.525	26.326
		0.1	0.026	25.206	71.795	83.077741289	71.376	50.879	32.231	19.099

Table 13: Shown Probability of A Voiding Second Sample W.R.T. Δ, U, N_1 and ζ

			ζ						
n_1	Δ	0.25	0.50	0.75	1	1.25	1.50	1.75	2
	0.01	0.717	0.952	0.986	0.990005	0.973	0.932	0.871	0.797
4	0.05	0.366	0.823	0.937	0.949994	0.907	0.828	0.733	0.635
	0.1	0.206	0.707	0.88	0.89997	0.841	0.744	0.637	0.537
	0.01	0.422	0.908	0.981	0.989999	0.967	0.908	0.816	0.709
6	0.05	0.128	0.719	0.921	0.949997	0.894	0.784	0.653	0.526
	0.1	0.052	0.576	0.854	0.90000	0.823	0.691	0.551	0.427
8	0.01	0.196	0.851	0.976	0.990000	0.962	0.882	0.76	0.623
	0.05	0.035	0.612	0.903	0.949991	0.881	0.74	0.579	0.432

	0.1	0.01	0.458	0.829	0.89998	0.806	0.642	0.475	0.337
	0.01	0.045	0.721	0.936	0.836	0.603	0.381	0.225	0.129
10	0.05	7.991E-3	0.51	0.885	0.949999	0.868	0.699	0.512	0.352
	0.1	1.807E-3	0.357	0.803	0.899990	0.79	0.595	0.408	0.265
	0.01	0.024	0.71	0.963	0.990002	0.95	0.827	0.649	0.467
12	0.05	1.59E-3	0.417	0.867	0.950002	0.855	0.658	0.45	0.285
	0.1	2.771E-4	0.273	0.778	0.900009	0.773	0.551	0.349	0.207

6. Conclusions

From the above discussions it is obvious that by using guess point value one can improve the minimax estimator by using shrinkage technique. It can be noted that if the guess point α_0 is very close to the true value of the parameter α (i.e.; ζ is approximate close to one), the proposed estimators perform better than the minimax estimator. If one has no confidence in the guessed value, then proposed preliminary test shrunken estimators can be suggested. We can safely use the proposed estimators for small sample size at the usual level of significance Δ and moderate value of shrunken weight factor $\Psi(.)$. The difficulty of obtaining samples because of the scarcity and high cost led researchers to use the double stage shrinkage estimators to reduce the sample size that we need and for achieving savings of the items in the sample and obtaining high efficiency estimators.

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